C O L L O Q U I U M   M A T H E M A T I C U M

VOL. * 200* NO.

THE BASS CONJECTURE AND GROWTH IN GROUPS

BY

ANDERS KARLSSON (Stockholm) and MARKUS NEUHAUSER (Graz)

Abstract. We discuss Bass's conjecture on the vanishing of the Hattori–Stallings rank from the viewpoint of geometric group theory. It is noted that groups without u-elements satisfy this conjecture. This leads in particular to a simple proof of the conjecture in the case of groups of subexponential growth.

1. Introduction. We show that a theorem recently formulated by Eckmann [E 01, Th. 9] allows a significant generalization. From this one can conclude for example that the class of groups of subexponential growth satisfies the Bass conjecture. These groups are not covered by [E 01, Th. 9], but they are known to satisfy the Bass conjecture from the more general result of the recent paper by Berrick, Chatterji, and Mislin [BCM02]. The proof of the latter result however is far from trivial, in particular it relies on some deep work of V. Lafforgue.

In Section 2 we emphasize that there are different kinds of infinite order elements. Indeed, it is natural to look at the behavior of the word length of powers of elements as a generalization of the order of finite order elements. In the literature this is usually referred to as "distortion in groups" ([G 93], [O 97], [P 02]). The observation on the Bass conjecture (Section 4) and on homomorphisms (Section 2) are applications of these considerations.

Although the ideas in this paper are simple and known to various groups of people, it is perhaps less clear how many are aware of all the present arguments and statements.

2. Growth of individual group elements. Let G be a finitely generated group and $\|\cdot\|$ the word length corresponding to a finite set of generators, i.e. the minimum length of a word in the generators representing an element g (see e.g. [dlH00]). For an element $g \in G$, it is natural to look at the behavior of the power length function

$$a_g : n \mapsto \|g^n\|.$$

2000 *Mathematics Subject Classification*: 16S34, 20E07, 20F65, 22D15.

The authors gratefully acknowledge the financial support of research grant 20-65060.01 of the Swiss National Fund for Scientific Research.

Partly because this function contains too much information and partly because it depends on the generating set (although in a mild way), it is convenient to introduce certain derivative notions such as:

- the *order* $o(g) = \min\{n \geq 1 : a_g(n) = 0\}$,
- the *translation length* $\tau(g) = \lim_{n\to\infty} a_g(n)/n$ (see [GS91]),
- *u-elements*: these are the infinite order elements g for which there is a constant $C > 0$ such that
$$a_g(n) \leq C \log n$$
for all $n > 1$ (cf. [LMR00]).

Note that the function $g \mapsto a_g$ is almost conjugation invariant. Therefore from various derivative notions one can obtain conjugation invariant functions on the group and traces on the group algebra.

Another fundamental conjugation invariant function is what in geometry is called the *minimum displacement function* $\delta(g)$: Given an isometric action of G on a metric space (X, d) (for example G itself with a word metric), one defines
$$\delta(g) = \inf_{x\in X} d(gx, x).$$
This corresponds algebraically to the minimum word length of the elements in the conjugacy class of g. Also with this function one can define a trace on the group algebra and it is not very far from the traces for example employed in [F 73].

The main property of a_g is that it semi-decreases under homomorphisms (just like the order for finite order elements); more precisely, given a homomorphism $\varphi : G \to H$, where H is equipped with an invariant metric d (such as a word metric), there is a constant C such that for all $g \in G$,
$$\|\varphi(g)\|_H \leq C\|g\|,$$
where $\|\cdot\|_H = d(\cdot, 1)$. Hence $a_{\varphi(g)} \leq C a_g$.

An application to $\mathrm{SL}(m, \mathbb{Z})$

THEOREM 1. *Any homomorphism from* $\mathrm{SL}(m, \mathbb{Z})$, $m \geq 3$, *into a group of subexponential growth* (*or a semi-hyperbolic group*, *or the mapping class group of a surface*, *etc.*) *must have finite image.*

Proof. It is known that the elementary matrices are u-elements (see e.g. [LMR00]). Moreover, from [CK84] we know that $\mathrm{SL}(m, \mathbb{Z})$ is boundedly generated by these elements, i.e. there exists an N (depending on m) such that every matrix can be written as a product of at most N powers of elementary matrices.

Since the target group has no u-elements (see below), it follows from the main property of a_g that the elementary matrices must map to finite order elements. The image consists of products of at most N powers of these finite order elements and hence the whole image group is finite. (Instead, one could appeal to Margulis's normal subgroup theorem, as for example in [P 02].) ∎

3. Groups with or without u-elements. As pointed out in [E 01], semi-hyperbolic groups (e.g. biautomatic groups: F_n, word hyperbolic groups, braid groups; CAT(0)-groups: $\mathbb{Z}^m$, Coxeter groups) have the property that a_g is either bounded or of linear growth for each element g (due to Alonso and Bridson in this generality). Let us also mention that groups of isometries of a Busemann non-positively curved space (e.g. the symmetric space associated to a C^*-algebra) for which $\delta(g) > 0$ for every $g \neq 1$, mapping class groups ([FLM01]) and certain groups of symplectic diffeomorphisms ([P 02]) also do not contain u-elements.

So-called Baumslag–Solitar relations give rise to u-elements:

LEMMA. *Assume that $g \in G$ is an infinite order element such that $g = bg^k b^{-1}$ for some $k > 1$ and $b \in G$. Then g is a u-element.*

Proof. Given $n > 1$, write it as $n = \sum_{i=0}^m c_i k^i$ where $0 \le c_i \le k-1$ and $m = \lfloor \log n / \log k \rfloor$. Now we have

$$\begin{aligned} \|g^n\| &= \|g^{c_0} g^{c_1 k} \cdots g^{c_m k^m}\| = \|g^{c_0} b^{-1} g^{c_1} b \cdots b^{-i} g^{c_i} b^i \cdots b^{-m} g^{c_m} b^m\| \\ &\le \|g\| \sum_{i=0}^m c_i + 2\|b\| m \le \left(\frac{\log n}{\log k} + 2\right)((k-1)\|g\| + 2\|b\|). \ \blacksquare \end{aligned}$$

On the other hand, any u-element has finite order in every quotient $G/G_{(n)}$, where $G_{(n)}$ are the subgroups in the lower central series of G. Indeed, a_g semi-decreases under homomorphisms and nilpotent groups have no u-elements (as follows from the next proposition).

PROPOSITION. *If G contains a u-element, then it is of exponential growth.*

Proof. Since a u-element g is of infinite order, all the g^n are distinct. Therefore the number of elements of length at most r is at least $e^{r/C}$ for some $C > 0$. ∎

4. Bass and idempotent conjectures. By the $\mathbb{C}G$-Bass conjecture we mean the assertion that the Hattori–Stallings rank r_P, which is a certain conjugacy invariant function $G \to \mathbb{C}$ associated to a finitely generated projective $\mathbb{C}G$-module P, vanishes on every infinite order element. We refer the reader to [B 76], [E 01], and [BCM02] for more information. In [B 76] Bass established his conjecture for linear groups and showed that the conjecture implies the idempotent conjecture for torsion-free groups, which asserts that the only elements p in $\mathbb{C}G$ such that $p^2 = p$ are $p = 0$ and $p = 1$. Moreover, in that paper it is explained that if $r_P(g) \neq 0$ for an infinite order element g, then g is conjugate to g^{p^n} for some $n > 0$ and almost all primes p. (Similar divisibility conditions appear in an earlier work of Formanek [F 73] in the context of the idempotent conjecture.) In view of this and the Lemma above, we have the following sharpened formulation of Theorem 9 in Eckmann's paper [E 01]:

THEOREM 2. *The $\mathbb{C}G$-Bass conjecture holds for every finitely generated group without u-elements.* (*If the group is in addition torsion-free the idempotent conjecture holds as well.*)

In view of the Proposition we hence have:

COROLLARY. *Every group of subexponential growth satisfies the $\mathbb{C}G$-Bass conjecture* (*and the idempotent conjecture as well provided the group is torsion-free*).

Note that every group of subexponential growth is amenable and that it was already established by Berrick, Chatterji, and Mislin in their recent paper [BCM02] that any amenable group satisfies Bass's conjecture. Indeed, they prove that the Bost conjecture implies (a stronger version of) the Bass conjecture and then they rely on results of V. Lafforgue on the Bost conjecture. Somewhat earlier the class of all elementary amenable groups (which in particular excludes the groups of subexponential but superpolynomial growth) was treated by Farrell and Linnell. To further put things in perspective we mention that $\mathrm{BS}(1,k) = \langle g, b : g = bg^k b^{-1} \rangle$ is an amenable group containing a u-element, and that groups of intermediate growth (first constructed by Grigorchuk in the early 1980s) are not linear.

The various divisibility conditions in [B 76] are interesting in their own right: what can be said for groups acting properly by isometry on a locally compact space of non-positive curvature?

REFERENCES

[B 76] H. Bass, *Euler characteristic and characters of discrete groups*, Invent. Math. 35 (1976), 155–196.

[BCM02] A. J. Berrick, I. Chatterji and G. Mislin, *From acyclic groups to the Bass conjecture for amenable groups*, preprint, FIM, ETH Zürich, 2002.

[CK84] D. Carter and G. Keller, *Elementary expressions for unimodular matrices*, Comm. Algebra 12 (1984), 379–389.

[E 01] B. Eckmann, *Idempotents in a complex group algebra, projective modules, and the von Neumann algebra*, Arch. Math. (Basel), 76 (2001), 241–249.

[FLM01] B. Farb, A. Lubotzky and Y. Minsky, *Rank-*1 *phenomena for mapping class groups*, Duke Math. J. 106 (2001), 581–597.

[F 73] E. Formanek, *Idempotents in Noetherian group rings*, Canad. J. Math. 25 (1973), 366–369.

[GS91] S. M. Gersten and H. B. Short, *Rational subgroups of biautomatic groups*, Ann. of Math. 134 (1991), 125–158

[G 93] M. Gromov, *Asymptotic invariants of infinite groups*, in: Geometric Group Theory, Vol. 2, G. A. Niblo and M. A. Roller (eds.), London Math. Soc. Lecture Note Ser. 182, Cambridge Univ. Press, Cambridge, 1993, 1–295.

[dlH00] P. de la Harpe, *Topics in Geometric Group Theory*, Chicago Lectures in Math., Univ. of Chicago Press, Chicago, 2000.

[LMR00] A. Lubotzky, S. Mozes and M. S. Raghunathan, *The word and Riemannian metrics on lattices of semisimple groups*, Inst. Hautes Études Sci. Publ. Math. 91 (2000), 5–53.

[O 97] A. Yu. Ol′shanskiĭ, *On subgroup distortion in finitely presented groups*, Sb. Math. 188 (1997), 1617–1664.

[P 02] L. Polterovich, *Growth of maps, distortion in groups and symplectic geometry*, Invent. Math. 150 (2002), 655–686.

Department of Mathematics
Royal Institute of Technology
100 44 Stockholm, Sweden
E-mail: akarl@math.kth.se

Department of Mathematics C
TU Graz
Steyrergasse 30/III
8010 Graz, Austria
E-mail: neuhauser@finanz.math.tu-graz.ac.at

Received 23 September 2003 (4379)