

Complexity and heights of tori

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ABSTRACT. We prove detailed asymptotics for the number of spanning trees, called complexity, for a general class of discrete tori as the parameters tend to infinity. The proof uses in particular certain ideas and techniques from an earlier paper [CJK10]. Our asymptotic formula provides a link between the complexity of these graphs and the height of associated real tori, and allows us to deduce some corollaries on the complexity thanks to certain results from analytic number theory. In this way we obtain a conjectural relationship between complexity and regular sphere packings.

1. Introduction

The number of spanning trees $\tau(G)$, called *the complexity*, of a finite graph G is an invariant which is of interest in several sciences: network theory, statistical physics, theoretical chemistry, etc. Via the well-known matrix-tree theorem of Kirchoff, the complexity equals the determinant of the combinatorial Laplacian Δ_G divided by the number of vertices.

For compact Riemannian manifolds M there is an analogous invariant $h(M)$, *the height*, defined as the negative of the logarithm of the zeta-regularized determinant of the Laplace-Beltrami operator, and which is of interest for quantum physics. The analogy between the height and complexity has been commented on by Sarnak in [S90].

In statistical physics it is of interest to study the asymptotics of the complexity, and other spectral invariants, for certain families of graphs. Important cases to study are various subgraphs of the standard lattice \mathbb{Z}^d . An instance of this is to study discrete tori, corresponding to periodic boundary conditions, as the parameters tend to infinity, see [DD88], [CJK10], and references therein. It is shown in [CJK10] that in the asymptotics of the complexity of discrete tori, the height of an associated real torus appears as a constant.

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In the present paper we study discrete tori of a more general type, defined as follows. Let Λ be an invertible $r \times r$ matrix with all entries being integers. This matrix defines a lattice $\Lambda\mathbb{Z}^r$ in \mathbb{R}^r . We associate to the group quotient with standard generators

$$\Lambda\mathbb{Z}^r \backslash \mathbb{Z}^r$$

its Cayley graph, which we call a *discrete torus*. In other words, two elements x and y in $\Lambda\mathbb{Z}^r \backslash \mathbb{Z}^r$ are *adjacent*, denoted $x \sim y$, if they differ by ± 1 in exactly one of the coordinates and equal everywhere else (everything mod $\Lambda\mathbb{Z}^r$ of course).

Let $0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_{|\det \Lambda| - 1}$ be the eigenvalues of the combinatorial Laplacian $\Delta_{\Lambda\mathbb{Z}^r \backslash \mathbb{Z}^r}$ of the discrete torus – see (2.1) for a definition of $\Delta_{\Lambda\mathbb{Z}^r \backslash \mathbb{Z}^r}$. Define $\det^* \Delta_{\Lambda\mathbb{Z}^r \backslash \mathbb{Z}^r}$ to be the product of the nonzero eigenvalues of the Laplacian:

$$\det^* \Delta_{\Lambda\mathbb{Z}^r \backslash \mathbb{Z}^r} := \lambda_1 \lambda_2 \dots \lambda_{|\det \Lambda| - 1}.$$

Note that the trivial eigenvalue is removed. We will nevertheless refer to $\det^* \Delta_{\Lambda\mathbb{Z}^r \backslash \mathbb{Z}^r}$ as the determinant of the Laplacian. For simplicity we will mostly assume that $\det \Lambda > 0$.

THEOREM 1.1. *Let $\{\Lambda_n\}$ be a sequence of $r \times r$ integer matrices. Suppose that $\det \Lambda_n \rightarrow \infty$ and $\Lambda_n / (\det \Lambda_n)^{1/r} \rightarrow A \in SL_r(\mathbb{R})$. Then as $n \rightarrow \infty$,*

$$\log \det^* \Delta_{\Lambda_n \mathbb{Z}^r \backslash \mathbb{Z}^r} = c_r \det \Lambda_n + \frac{2}{r} \log \det \Lambda_n + \log \det^* \Delta_{A\mathbb{Z}^r \backslash \mathbb{R}^r} + o(1)$$

where

$$c_r = \log 2r - \int_0^\infty e^{-2rt} (I_0(2t)^r - 1) \frac{dt}{t}.$$

The definition of $\log \det^* \Delta_{A\mathbb{Z}^r \backslash \mathbb{R}^r}$ will be recalled in section 3 below.

Our earlier paper [CJK10] treats the case when the Λ_n are diagonal matrices. The present paper uses several facts that are established in that paper. Thanks to the fact that the first two terms in the asymptotics are universal in the sense that they only depend on Λ_n via $\det \Lambda_n$, the theorem gives a close connection between the complexity of certain graphs and the height of an associated manifold. We emphasize that this is not at all obvious: while it is true that the appropriately rescaled eigenvalues of the discrete tori converge to the eigenvalues of $\Delta_{A\mathbb{Z}^r \backslash \mathbb{R}^r}$, this convergence is certainly not uniform. Moreover, the height cannot be defined as the product of eigenvalues, there is a regularization in the definition. An explicit expression for the height of flat tori of volume 1, $h(A\mathbb{Z}^r \backslash \mathbb{R}^r) := -\log \det \Delta_{A\mathbb{Z}^r \backslash \mathbb{R}^r}$, can be found in Theorem 2.3 of [Ch97]. Deninger and Lück have informed us that the constant c_r also has an interpretation as a determinant, namely the Fuglede-Kadison determinant of the Laplacian on \mathbb{Z}^r , see [L].

We turn now to a connection between our results and sphere packings. The problem of finding the densest packing of ordinary space with spheres of equal radii is an old one with practical importance even in dimensions greater than 3. One type of packings is *regular sphere packings* which means that the spheres are centered at the points of a lattice $A\mathbb{Z}^r$. Gauss showed that the face centered lattice (fcc) D_3 is optimal among regular packings in three dimensions. By work of Thue and Toth one knows that the hexagonal lattice A_2 is densest in dimension 2. In fact, Thue and Toth proved that A_2 gives the densest packing among all packings — lattice as well as non-lattice. In dimension 24 it is known that the Leech lattice is optimal

among unimodular lattices. We refer to the book of Conway-Sloane [CS99] and the paper of Cohn-Kumar [CK09] for more information.

Conjecturally the height of $A\mathbb{Z}^r \setminus \mathbb{R}^r$ has a global minimum when $A\mathbb{Z}^r$ is the densest regular sphere packing. This conjecture, attributed to Sarnak, appears as Conjecture 4.5 in [Ch97]. Extremal metrics for heights has been studied in [OPS88] in dimension 2, and for tori in higher dimensions notably in [Ch97] and [SS06]. In these papers, the question is phrased as the study of the derivative of Epstein zeta functions at $s = 0$. From this theory we can deduce the following corollaries from our main theorem:

COROLLARY 1.2. *Let $\{\Lambda_n\}$ be a sequence of $r \times r$ integer matrices with $\det \Lambda_n \rightarrow \infty$. Suppose that $\{\Lambda_n / (\det \Lambda_n)^{1/r}\}$ belongs to a compact subset of $SL_r(\mathbb{R})$, $r = 2, 3$, avoiding lattices equivalent to A_2 , resp. D_3 . Assume that there is a sequence $\{L_n\}$ with $\det L_n = \det \Lambda_n$ such that $\{L_n / (\det L_n)^{1/r}\}$ converges to A_2 , resp. D_3 . Then $L_n \mathbb{Z}^r \setminus \mathbb{Z}^r$ has more spanning trees than $\Lambda_n \mathbb{Z}^r \setminus \mathbb{Z}^r$ for all sufficiently large n .*

COROLLARY 1.3. *Let Λ_n be a sequence of $r \times r$ integer matrices with $\det \Lambda_n \rightarrow \infty$. Suppose that $\{\Lambda_n / (\det \Lambda_n)^{1/r}\}$ stays in a compact subset of $SL_r(\mathbb{R})$. For all sufficiently large n we have that*

$$\tau(\Lambda_n \mathbb{Z}^r \setminus \mathbb{Z}^r) \leq \frac{(\det \Lambda_n)^{2/r-1}}{4\pi} \exp(c_r \det \Lambda_n + \gamma + 2/r),$$

where γ is Euler's constant and c_r is as in the theorem.

In the trivial case $r = 1$, this estimate gives a value close to the truth:

$$\det \Lambda_n = \tau(\Lambda_n \mathbb{Z} \setminus \mathbb{Z}) \leq 1.05 \det \Lambda_n.$$

Upper bounds for the number of spanning trees have been considered in the combinatorics literature since 1970s at least. For regular graphs there is a rather sharp estimate by Chung and Yau [CY99] improving on an earlier result of McKay [M]. In general, it is an open problem to decide which simple graph on n vertices and e edges has the maximal complexity. This is of interest to communication network theory since this graph invariant appears as a measure of reliability.

It would be of interest to also go in the other direction: proving results on the extrema of families of Epstein zeta functions via a better understanding of the number of spanning trees of discrete tori.

2. Spectral preliminaries for discrete tori

Let Λ be an invertible $r \times r$ integer matrix. This matrix defines a lattice $\Lambda\mathbb{Z}^r$ in \mathbb{R}^r . We denote by $DT(\Lambda)$ the *discrete torus*, or Cayley graph of the quotient group $\Lambda\mathbb{Z}^r \setminus \mathbb{Z}^r$ with standard generating set: two elements x and y in $\Lambda\mathbb{Z}^r \setminus \mathbb{Z}^r$ are *adjacent*, denoted $x \sim y$, if they differ by ± 1 in exactly one of the coordinates and equal everywhere else (everything mod $\Lambda\mathbb{Z}^r$ of course).

The associated (combinatorial) *Laplacian* is defined by

$$(2.1) \quad \Delta_{DT(\Lambda)} f(x) = \sum_{y \text{ s.t. } y \sim x} (f(x) - f(y))$$

on functions $f : \Lambda\mathbb{Z}^r \setminus \mathbb{Z}^r \rightarrow \mathbb{R}$.

The *dual lattice* $\Lambda^* \mathbb{Z}^r$ is as usual all the points v in \mathbb{R}^r such that $(v, x) \in \mathbb{Z}$ for all $x \in \Lambda\mathbb{Z}^r$, where (\cdot, \cdot) denotes the usual scalar product. Since \mathbb{Z}^r is self-dual and

$\Lambda\mathbb{Z}^r$ is a subgroup of \mathbb{Z}^r it follows that \mathbb{Z}^r is a subgroup of $\Lambda^*\mathbb{Z}^r$. Note that the respective indices are

$$[\mathbb{Z}^r : \Lambda\mathbb{Z}^r] = [\Lambda^* : \mathbb{Z}^r] = |\det \Lambda|.$$

PROPOSITION 2.1. *The eigenfunctions of $\Delta_{DT(\Lambda)}$ are given by*

$$f_v(x) = e^{2\pi i(x,v)},$$

for each $v \in \mathbb{Z}^r \setminus \Lambda^*\mathbb{Z}^r$, with corresponding eigenvalue given by

$$\lambda_v = 2r - 2 \sum_{k=1}^r \cos(2\pi v_k),$$

where v_k denotes the k th coordinate of v .

PROOF. The operator $\Delta_{DT(\Lambda)}$ is a semipositive, symmetric matrix and hence we are looking for $|\det \Lambda|$ number of eigenfunctions and eigenvalues. The proof is a trivial calculation:

$$\begin{aligned} \Delta e^{2\pi i(x,v)} &= 2r e^{2\pi i(x,v)} - \sum_{y \sim x} e^{2\pi i(y,v)} = \\ &= \left(2r - \sum_{z \sim 0} e^{2\pi i(z,v)} \right) e^{2\pi i(x,v)}. \end{aligned}$$

□

The heat kernel $K^\Lambda(t, x) : \mathbb{R}_{\geq 0} \times DT(\Lambda) \rightarrow \mathbb{R}$ is the unique bounded function which satisfies

$$\begin{aligned} \left(\Delta_{DT(\Lambda)} + \frac{\partial}{\partial t} \right) K^\Lambda(t, x) &= 0 \\ K^\Lambda(0, x) &= \delta_0(x), \end{aligned}$$

where $\delta_0(x) = 1$ if $x = 0$ and 0 otherwise. The existence and uniqueness of heat kernels in a general graph setting is established in [DM06]. Recall from e.g. [CJK10] that

$$K^{\mathbb{Z}^r}(t, z) = \prod_{k=1}^r K^{\mathbb{Z}}(t, z_k)$$

for $z = (z_k)$ and $K^{\mathbb{Z}}(t, w) = e^{-2t} I_w(2t)$, where I_w is the I -Bessel function of order w .

We have the following *theta inversion formula* (cf. [CJK10]).

PROPOSITION 2.2. *The following formula holds for $x \in \Lambda\mathbb{Z}^r \setminus \mathbb{Z}^r$ and $t \in \mathbb{R}_{\geq 0}$*

$$\sum_{y \in \Lambda\mathbb{Z}^r} K^{\mathbb{Z}^r}(t, x - y) = \frac{1}{|\det \Lambda|} \sum_{\nu \in \mathbb{Z}^r \setminus \Lambda^*\mathbb{Z}^r} e^{-t\lambda_\nu} f_\nu(x).$$

In particular,

$$\theta_\Lambda(t) := |\det \Lambda| \sum_{y \in \Lambda\mathbb{Z}^r} e^{-2rt} I_{y_1}(2t) \dots I_{y_r}(2t) = \sum_{\nu \in \mathbb{Z}^r \setminus \Lambda^*\mathbb{Z}^r} e^{-t\lambda_\nu}$$

PROOF. Since both sides of the equation satisfy the conditions for being the heat kernel, this follows from the uniqueness of heat kernels. The second formula is the special case $x = 0$. □

3. Spectral preliminaries for continuous tori

An r -dimensional (continuous) torus is given as a quotient of \mathbb{R}^r by a lattice $A\mathbb{Z}^r$ where $A \in GL_r(\mathbb{R})$. The metric structure and the standard (positive) Laplace-Beltrami operator $-\sum_i \partial^2 / \partial x_i^2$ on \mathbb{R}^r projects to the torus. The volume is $|\det A|$. Let A^* be the matrix defining the dual lattice $A^*\mathbb{Z}^r$, and so $A^* = (A^{-1})^t$. The eigenfunctions of the Laplacian on the torus in question are $f_v(x) = \exp(2\pi i v^t x)$ where v are the vectors in the dual lattice. The corresponding eigenvalues are $\lambda_v = 4\pi^2 \|v\|^2$ or with a different indexing: $\lambda_m = (2\pi)^2 (A^*m)^t (A^*m)$, where m runs through \mathbb{Z}^r . We have the associated theta function

$$\Theta_A(t) = \sum_{m \in \mathbb{Z}^r} e^{-(2\pi)^2 (A^*m)^t (A^*m) \cdot t}.$$

The theta inversion formula, which is equivalent to the Poisson summation formula in this case, yields

$$\Theta_A(t) = \frac{1}{(4\pi t)^{r/2}} \sum_{x \in A\mathbb{Z}^r} e^{-|x|^2 / 4t}.$$

The associated spectral zeta function, which in this case also goes under the name of the Epstein zeta function, is defined as

$$Z_A(s) = \sum_{m \neq 0} \lambda_m^{-s} = \frac{1}{(2\pi)^{2s}} \sum_{v \neq 0} \frac{1}{\|v\|^{2s}}.$$

Classically, one can prove the meromorphic continuation of $Z_A(s)$ to all $s \in \mathbb{C}$, showing that its continuation is holomorphic at $s = 0$. From this, one defines the spectral determinant $\det^* \Delta_{A\mathbb{Z}^r \setminus \mathbb{R}^r}$ by

$$\log \det^* \Delta_{A\mathbb{Z}^r \setminus \mathbb{R}^r} = -Z'_A(0).$$

4. Asymptotics

Let

$$\mathcal{I}_r(s) = - \int_0^\infty \left(e^{-s^2 t} e^{-2rt} I_0(2t)^r - e^{-t} \right) \frac{dt}{t}$$

and

$$\mathcal{H}_\Lambda(s) = - \int_0^\infty \left(e^{-s^2 t} [\theta_\Lambda(t) - |\det \Lambda| \cdot e^{-2rt} I_0(2t)^r - 1] + e^{-t} \right) \frac{dt}{t}.$$

Everything in section 3 of [CJK10] carries over with only notational changes, even though the eigenvalues are different and the theta identity is hence somewhat different. These differences are not essentially used. In particular the first order term as $t \rightarrow 0$ in the trace of the heat kernel is still (in the present notation) $|\det \Lambda| \cdot e^{-2rt} I_0(2t)^r$ since it corresponds to the trivial eigenvalue. In particular the following extension of Theorem 3.6 in [CJK10] holds:

THEOREM 4.1. *For any $s \in \mathbb{C}$ with $\operatorname{Re}(s^2) > 0$, we have the relation*

$$\sum_{\lambda_v \neq 0} \log(s^2 + \lambda_v) = |\det \Lambda| \cdot \mathcal{I}_r(s) + \mathcal{H}_\Lambda(s).$$

Letting $s \rightarrow 0$ we have the identity

$$\log \left(\prod_{\lambda_v \neq 0} \lambda_v \right) = |\det \Lambda| \cdot \mathcal{I}_r(0) + \mathcal{H}_\Lambda(0).$$

Section 4 of [CJK10] is an independent section on uniform bounds on I -Bessel functions. We recall the following statements, slightly adapted to the present context (keeping in mind that $I_{-y} = I_y$ for integers, and that b may now be real):

PROPOSITION 4.2. *For any $t > 0$ and $b \geq 0$, there is a constant C such that*

$$0 \leq \sqrt{b^2 t} e^{-b^2 t} I_0(b^2 t) \leq C < 1$$

Fix $t \geq 0$ and integers $y, n_0 \geq 0$. Then for all $b \geq n_0$ we have the uniform bound

$$0 \leq \sqrt{b^2 t} \cdot e^{-b^2 t} I_y(b^2 t) \leq \left(1 + \frac{y}{bn_0 t}\right)^{-n_0 y / 2b}.$$

PROPOSITION 4.3. *Let $N(u)$ be a sequence of positive integers parametrized by $u \in \mathbb{Z}_+$ such that $N(u)/u \rightarrow \alpha > 0$ as $u \rightarrow \infty$. Then for any $t > 0$ and integer k , we have*

$$\lim_{u \rightarrow \infty} N(u) e^{-2u^2 t} I_{N(u)k}(2u^2 t) = \frac{\alpha}{\sqrt{4\pi t}} e^{-(\alpha k)^2 / 4t}.$$

From now on we fix a sequence $\{\Lambda_n\}$ of integer matrices with $0 < \det \Lambda_n \rightarrow \infty$ satisfying

$$\frac{1}{(\det \Lambda_n)^{1/r}} \Lambda_n \rightarrow A \text{ as } n \rightarrow \infty,$$

for some $A \in \mathrm{SL}_r(\mathbb{R})$. From the previous propositions we will deduce the following:

PROPOSITION 4.4. *For each fixed $t > 0$ we have the pointwise convergence*

$$\theta_{\Lambda_n}(\det(\Lambda_n)^{2/r} t) \rightarrow \theta_A(t)$$

as $n \rightarrow \infty$.

PROOF. For any $v \in \mathbb{Z}^r$ and $\Lambda \in \mathrm{GL}_r(\mathbb{R})$, let

$$\mathbf{I}_{v, \Lambda}(t) = \prod_{i=1}^r I_{(\Lambda v)_i}(t)$$

where $(\Lambda v)_i$ denotes the i -th component of Λv . Let $u_n = \det(\Lambda_n)^{1/r}$ and $a_i = (Av)_i$. Note that $(\Lambda_n v)_i / u_n \rightarrow a_i$. We have

$$\theta_{\Lambda_n}(u_n^2 t) = \sum_{v \in \mathbb{Z}^r} u_n^r e^{-2ru_n^2 t} \mathbf{I}_{v, \Lambda_n}(2u_n^2 t).$$

From Proposition 4.3 (with $k = 0$ or ± 1) we have for any $t > 0$ and $v \in \mathbb{Z}^r$ that

$$u_n^r e^{-2ru_n^2 t} \mathbf{I}_{v, \Lambda_n}(2u_n^2 t) \rightarrow \frac{1}{(\sqrt{4\pi t})^r} e^{-a_1^2 / 4t} \dots e^{-a_r^2 / 4t}$$

as $n \rightarrow \infty$. This means that the proposition will be proved if we can interchange the limit and the infinite sum. We show that for fixed t , the sum is convergent uniformly in u_n (or equivalently, in n).

We can rewrite the sum $\theta_{\Lambda_n}(u_n^2 t)$ in $r + 1$ sums depending on how many components of the $\Lambda_n v$ are zero. Pick n_0 sufficiently large so that

$$|a_i| / 2 \leq |(\Lambda_n v)_i| / u_n \leq 2 |a_i|$$

for all $v \in \mathbb{Z}^r$ and $n \geq n_0$ and $a_i \neq 0$. Recall that $I_{-n} = I_n$. Let us look at a term with k zeros in the y_i s and estimate with the help of Proposition 4.2:

$$\begin{aligned} u_n^r e^{-2ru_n^2 t} \mathbf{I}_{v, \Lambda_n}(2u_n^2 t) &\leq \left(\frac{1}{\sqrt{2t}}\right)^r \prod_{(\Lambda_n v)_i \neq 0} \left(1 + \frac{|(\Lambda_n v)_i|}{u_n n_0 2t}\right)^{-n_0 |(\Lambda_n v)_i| / 2u_n} \\ &\leq \left(\frac{1}{\sqrt{2t}}\right)^r \prod_{a_i \neq 0} \lambda^{|a_i|} \end{aligned}$$

for all n large and where

$$\lambda := \left(1 + \frac{a}{n_0 4t}\right)^{-n_0/4} < 1$$

and a is the smallest nonzero absolute value of all the entries in $A\mathbb{Z}^r$. The whole theta series is therefore bounded by $r+1$ sums of a product of convergent geometric series. This shows that the infinite sum is uniformly convergent and the proof is complete. \square

LEMMA 4.5. *Given a sequence $\{\Lambda_n\}$ satisfying $(\det \Lambda_n)^{-1/r} \Lambda_n \rightarrow A$ as above, there is a constant $d > 0$ such that for all sufficiently large n*

$$\theta_{\Lambda_n}(u_n^2 t) \leq 1 + \sum_{j=1}^{\infty} e^{-dtj},$$

for all $t > 0$.

PROOF. Let $u_n = \det(\Lambda_n)^{1/r}$. We have

$$\begin{aligned} \theta_{\Lambda_n}(t) &= \sum_{\nu \in \mathbb{Z}^r \setminus \Lambda_n^* \mathbb{Z}^r} e^{-t\lambda_\nu} = \sum_{\nu \in \mathbb{Z}^r \setminus \Lambda_n^* \mathbb{Z}^r} e^{-t(2r-2\sum_{k=1}^r \cos(2\pi v_k))} \\ &= 1 + \sum_{\substack{\nu \in \mathbb{Z}^r \setminus \Lambda_n^* \mathbb{Z}^r \\ \nu \neq 0}} \prod_{k=1}^r e^{-4t \sin^2(\pi v_k)}, \end{aligned}$$

so that

$$\theta_{\Lambda_n}(u_n^2 t) = 1 + \sum_{\substack{\nu \in \mathbb{Z}^r \setminus \Lambda_n^* \mathbb{Z}^r \\ \nu \neq 0}} \prod_{k=1}^r e^{-4tu_n^2 \sin^2(\pi v_k)}.$$

We use the elementary bounds $\sin x \geq x - x^3/6$ and $\sin(\pi - x) \geq x - x^3/6$ for $x \in [0, \pi/2]$ and get

$$\begin{aligned} u_n \sin(\pi v_k) &\geq u_n \pi v_k (1 - \pi^2 v_k^2 / 6) > c\pi u_n v_k \text{ if } v_k \leq 1/2 \\ u_n \sin(\pi v_k) &\geq u_n \pi v_k (1 - \pi^2 v_k^2 / 6) > c\pi u_n (1 - v_k) \text{ if } v_k > 1/2 \end{aligned}$$

for some positive constant c for all n sufficiently large. Note also that for every $v \neq 0$, the values $u_n v_k$ range over the integers times an entry in A as $n \rightarrow \infty$ because of the convergence of $(\det \Lambda_n)^{-1/r} \Lambda_n$ to A in $\mathrm{SL}_r(\mathbb{R})$. We then conclude there is a constant $d > 0$ such that for all sufficiently large n

$$\theta_{\Lambda_n}(u_n^2 t) \leq 1 + \sum_{j=1}^{\infty} e^{-dtj}.$$

\square

Now we can show:

PROPOSITION 4.6. *With the notation as above and $u_n := \det(\Lambda_n)^{1/r}$, we have that*

$$\begin{aligned} & \int_1^\infty \left(\theta_{\Lambda_n}(u_n^2 t) - u_n^r e^{-2ru_n^2 t} I_0(2u_n^2 t)^r - 1 + e^{-u_n^2 t} \right) \frac{dt}{t} \\ &= \int_1^\infty (\theta_A(t) - 1) \frac{dt}{t} - \frac{2}{r} (4\pi)^{-r/2} + o(1) \end{aligned}$$

as $n \rightarrow \infty$.

PROOF. Write

$$\begin{aligned} & \int_1^\infty \left(\theta_{\Lambda_n}(u_n^2 t) - u_n^r e^{-2ru_n^2 t} I_0(2u_n^2 t)^r - 1 + e^{-u_n^2 t} \right) \frac{dt}{t} \\ &= \int_1^\infty (\theta_{\Lambda_n}(u_n^2 t) - 1) \frac{dt}{t} - \int_1^\infty u_n^r e^{-2ru_n^2 t} I_0(2u_n^2 t)^r \frac{dt}{t} + \int_1^\infty e^{-u_n^2 t} \frac{dt}{t}. \end{aligned}$$

In the last row, the third integral clearly goes to zero as $n \rightarrow \infty$. For the first integral in the same row we have

$$\int_1^\infty (\theta_{\Lambda_n}(u_n^2 t) - 1) \frac{dt}{t} \rightarrow \int_1^\infty (\theta_A(t) - 1) \frac{dt}{t}$$

in view of the pointwise convergence from Proposition 4.4 and the uniform integrable upper bound from Lemma 4.5.

The middle integral

$$\int_1^\infty u_n^r e^{-2ru_n^2 t} I_0(2u_n^2 t)^r \frac{dt}{t}$$

converges to

$$\int_1^\infty (4\pi t)^{-r/2} \frac{dt}{t} = \frac{2}{r} (4\pi)^{-r/2}$$

in view of the heat kernel convergence from Proposition 4.2 and Proposition 4.3, so then we may appeal to the Lebesgue dominated convergence theorem. \square

Next we show:

PROPOSITION 4.7. *With the notation as above and $u_n := \det(\Lambda_n)^{1/r}$, we have that*

$$\int_0^1 \left(\theta_{\Lambda_n}(u_n^2 t) - u_n^r e^{-2ru_n^2 t} I_0(2u_n^2 t)^r \right) \frac{dt}{t} \rightarrow \int_0^1 \left(\theta_A(t) - (4\pi t)^{-r/2} \right) \frac{dt}{t}$$

as $n \rightarrow \infty$.

PROOF. For fixed t we have the pointwise convergence as $n \rightarrow \infty$

$$\theta_{\Lambda_n}(u_n^2 t) - u_n^r e^{-2ru_n^2 t} I_0(2u_n^2 t)^r \rightarrow \theta_A(t) - (4\pi t)^{-r/2}.$$

It remains therefore to exhibit uniform (for $n \gg 1$) integrable bounds on the integrands. This can be done in the same way as in the proof of Proposition 4.4. In order to make the bound obtained there, in terms of λ , integrable for $0 \leq t \leq 1$, we just need to choose n_0 large so that $n_0 a/4 > r/2$. \square

Finally we recall from [CJK10]:

PROPOSITION 4.8. *For $u \in \mathbb{R}$ we have the asymptotic formula*

$$\int_0^1 (e^{-u^2 t} - 1) \frac{dt}{t} = \Gamma'(1) - 2 \log(u) + o(1)$$

as $u \rightarrow \infty$.

We now turn to the proof of our main result, Theorem 1.

In view of Theorem 4.1 we have

$$\log \det \Delta_{DT(\Lambda_n)} = \det \Lambda_n \cdot c_r - \int_0^\infty (\theta_{\Lambda_n}(t) - \det \Lambda_n \cdot e^{-2rt} I_0(2t)^r - 1 + e^{-t}) \frac{dt}{t}.$$

After the change of variables $t \rightarrow u_n^2 t$, the second term becomes

$$\begin{aligned} & - \int_0^\infty (\theta_{\Lambda_n}(u_n^2 t) - \det \Lambda_n \cdot e^{-2ru_n^2 t} I_0(2u_n^2 t)^r - 1 + e^{-u_n^2 t}) \frac{dt}{t} \\ & = - \left[\int_0^1 + \int_1^\infty \right] (\theta_{\Lambda_n}(u_n^2 t) - \det \Lambda_n \cdot e^{-2ru_n^2 t} I_0(2u_n^2 t)^r - 1 + e^{-u_n^2 t}) \frac{dt}{t}. \end{aligned}$$

In view of Propositions 4.6, 4.7, and 4.8, this integral equals

$$\begin{aligned} & - \int_1^\infty (\theta_A(t) - 1) \frac{dt}{t} + \frac{2}{r} (4\pi)^{-r/2} - \int_0^1 (\theta_A(t) - (4\pi t)^{-r/2}) \frac{dt}{t} \\ & - \Gamma'(1) + 2 \log(u_n) + o(1). \end{aligned}$$

Keeping in mind that $u_n = (\det \Lambda_n)^{1/r}$ and identifying the constant terms appearing in the meromorphic continuation of $-\zeta'(0) = \log \det \Delta_{AZ^r \setminus \mathbb{R}^r}$, the main theorem is proved; cf. equation (15) of [CJK10] with $V(A) = \det A = 1$.

5. Proof of the corollaries

To prove the corollaries in the introduction we recall the statements from the literature that we use.

For Corollary 1.2 note that the height has a global minimum for the hexagonal lattice A_2 in dimension 2 as is well-known and for the f.c.c. lattice $A_3 \cong D_3$ in dimension 3 by the rigorous numerics of Sarnak-Strömbergsson in [SS06]. Hence for any unimodular lattice L in the respective dimensions

$$\begin{aligned} h(L) &\geq h(A_2), \text{ so } \log \det \Delta_L \leq \log \det \Delta_{A_2} \\ h(L) &\geq h(D_3), \text{ so } \log \det \Delta_L \leq \log \det \Delta_{D_3}. \end{aligned}$$

As already remarked the two leading terms in the asymptotics in Theorem 1.1 are shape-independent, and so Corollary 1.2 follows from these remarks arguing with convergent subsequences in view of the compactness.

For Corollary 1.3 note that Corollary 1 on p. 119 in [SS06] implies that

$$\log \det \Delta_M < \gamma - \log 4\pi + \frac{2}{r} < -0.95$$

where M is an r -dimensional flat torus of volume 1 and γ is Euler's constant $\gamma \approx 0.577$. In view of this statement, let us replace Λ_n by a convergent subsequence. Then by Theorem 1.1 and the matrix-tree theorem, we have

$$\tau(DT(\Lambda_n)) = \frac{\det' \Delta_{DT(\Lambda_n)}}{\det \Lambda_n} \leq \frac{(\det \Lambda_n)^{2/r-1}}{4\pi} \exp(\det \Lambda_n \cdot c_r + \gamma + 2/r)$$

for all sufficiently large n . This concludes the proof of Corollary 1.3.

Finally, it may be of interest to mention another estimate in [SS06]:

$$h(L) \geq 4\sqrt{\frac{\pi}{r}} \left(\frac{\sqrt{r/2\pi e}}{m(L)} \right)^r (1 + o(1)), \text{ as } r \rightarrow \infty$$

where $m(L)$ is the length of the shortest non-zero vector in the lattice L . Recall that being the densest regular packing is equivalent to being the lattice with co-volume 1 which maximizes the length of the shortest nonzero vector.

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