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Free Subgroups of Groups with Nontrivial Floyd Boundary

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ABSTRACT

We prove that when a countable group admits a nontrivial Floyd-type boundary, then every nonelementary and metrically proper subgroup contains a noncommutative free subgroup. This generalizes the corresponding well-known results for hyperbolic groups and groups with infinitely many ends. It also shows that no finitely generated amenable group admits a nontrivial boundary of this type. This improves on a theorem by Floyd (Floyd, W. J. (1980). Group completions and limit sets of Kleinian groups. *Invent. Math.* 57: 205–218) as well as giving an elementary proof of a conjecture stated in that same paper. We also show that if the Floyd boundary of a finitely generated group is nontrivial, then it is a boundary in the sense of Furstenberg and the group acts on it as a convergence group.

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1. INTRODUCTION

For a group Γ generated by a finite set S, one may associate the Cayley graph $C(\Gamma, S)$ where the vertex set is the group itself and two vertices are connected by an edge if they differ by multiplication of an element in S on the left. Freudenthal and Hopf introduced the *end-compactification* of this graph as the graph (or the group) union the set of ends of this graph (Freudenthal, 1931, 1942). There is a natural topology and the compactification is independent of the finite generating set. Hopf (1944) showed that a finitely generated group has either 0, 1, 2, or uncountably many ends and that it has two ends if and only if it is virtually $\mathbb{Z}$. Stallings (1968) proved in a remarkable paper (treating the torsion-free case) that a finitely generated group has infinitely many ends if and only if it is an amalgamated free product $A *_C B$ or HNN-extension $A *_C$ with C finite, $|A/C| \geq 3$ and $|B/C| \geq 2$.

Most groups have only one end and so the end-compactification is trivial. Examples of one-ended groups include fundamental groups of compact hyperbolic manifolds. Floyd (1980) introduced a larger boundary which is obtained by rescaling the edge-path metric on $C(\Gamma, S)$ by a conformal factor of, for example, $d(e, g)^{-2}$ and then taking the metric completion. Floyd used this completion to study limit sets of Kleinian groups. See also Floyd (1984), Stark (1992) and Tukia (1988). Gromov (1987) discussed this type of boundary constructions in his essay on hyperbolic groups. In the case of word hyperbolic groups the Floyd completion, under some conditions on the conformal factor, is homeomorphic to the usual hyperbolic compactification.

Just as the end-compactification often is trivial, the Floyd boundary construction has a tendency to degenerate: the boundary of the product of two infinite groups is one point. (This can be avoided by for example allowing S to be infinite.) Even if the group is not a product, the Floyd boundary may degenerate in the presence of too many higher rank free abelian subgroups, as is the case for most mapping class groups, braid groups and $SL(n, \mathbb{Z})$ ($n \geq 3$), see Karlsson and Noskov (2002b).

A Floyd boundary is called *trivial* if it consists of 0, 1, or 2 points. For finitely generated groups, the property of admitting a nontrivial Floyd boundary (with respect to a finite S) is a quasi-isometric invariant. The class of groups with nontrivial Floyd boundary contains nonelementary word hyperbolic groups (see Gromov, 1987, 7.2.K), groups with infinitely many ends (a simple fact), and nonelementary geometrically finite Kleinian groups (see Floyd, 1980; Tukia, 1988).

Let Γ be a group generated by a countable (or finite) set S. The Floyd boundary $\partial\Gamma$ depends on S and a conformal factor f, see Sec. II. A subgroup Λ of Γ is said to be *nonelementary with respect to* $\partial\Gamma$ if there exists a sequence g_n in Λ such that both g_n and g_n^{-1} converge to points of $\partial\Gamma$ and Λ does not fix this or these limit point(s) setwise. Our main results are:

Theorem 1. *Let Γ be a group generated by a finite or countable set S and let Λ be a subgroup. Assume that Λ is nonelementary with respect to $\partial\Gamma$ and that every infinite subset of Λ is unbounded in (Γ, d). Then Λ contains a noncommutative free subgroup.*

We refer to the paper of Woess (1993) for a discussion of and references to previous results of this type. Compare also with Gromov (1993, Ch. 8). Furthermore, we have:

Theorem 2. *Assume that Γ is generated by a finite set S and that $\partial\Gamma$ is nontrivial. Then $\partial\Gamma$ is a boundary of Γ in the sense of Furstenberg* (1973)*. Furthermore, Γ acts on $\partial\Gamma$ as a convergence group.*

In order to apply Theorem 1 it is important to know criteria for the existence of convergent sequences g_n and for a subgroup to be nonelementary. In Sec. IV we obtain in particular:

Proposition 1. *Assume that g is an element of Γ such that $d(e, g^n) \to \infty$ as $n \to \infty$. Then both g^n and g^{-n} converge to points (or the same point) in $\partial\Gamma$.*

Proposition 2. *Let Γ be a group generated by a finite set S and let Λ be a subgroup. If the limit set of Λ contains at least three points in $\partial\Gamma$. Then Λ is nonelementary with respect to $\partial\Gamma$.*

Hence we have:

Corollary 1. *Assume that Γ is generated by a finite set S. If $\partial\Gamma$ contains at least three points, then Γ contains a noncommutative free subgroup.*

In view of the proof of Proposition 5 we have:

Corollary 2. *Let Γ be a group generated by a finite or countable set S and let Λ be a subgroup. Assume that every infinite subset of Λ is unbounded in (Γ, d) and that h is an element with two distinct limit points. If the group* *generated by h and another element g does not contain a noncommutative free subgroup, then* $h^l = gh^k g^{-1}$, *for some nonzero integers k and l.*

Amenable groups and torsion groups are among finitely generated groups which have no noncommutative free subgroups. Observe also that $\partial\Gamma$, when nontrivial, is a compact Γ-space without a Γ-invariant probability measure.

Corollary 3. *If* Γ *is a finitely generated amenable group or a finitely generated torsion group, then every Floyd boundary* $\partial\Gamma$ *is trivial.*

The class of amenable groups contains every virtually solvable group and every finitely generated group of subexponential growth. Therefore our corollary generalizes one of the main theorems in Floyd (1980) and shows the truth of a statement significantly stronger than a conjecture formulated in that same paper: Floyd proved that any finitely generated polycyclic group of one end must have trivial boundary and conjectured that every finitely generated group of polynomial growth has trivial boundary. The conjecture was of course settled as a consequence of Gromov's polynomial growth theorem. Corollary 3 also generalizes the fact that an infinite torsion group has only one end which was proved in Freudenthal (1945).

Here follow a few further remarks. Gromov (1987) wrote that "[t]he compactification of *any* Γ by [the space of ends] suggests a general notion of (partially) hyperbolic boundaries [...]. In particular one may seek a *maximal* hyperbolic boundary similar to the Furstenberg boundary (which is $\partial\Gamma$ if Γ is word hyperbolic)." As illustrated in the present paper the Floyd boundary is a hyperbolic-type boundary. Moreover, it was shown in Karlsson (2003) that when the Floyd boundary of Γ is nontrivial then it is maximal in the sense of Poisson boundaries of Γ (with respect to reasonable measures) and it is a mean proximal space in the sense of Furstenberg (1973).

In the recent article (McMullen, 2001), a conjecture is stated concerning the existence of a continuous surjective map from the Floyd boundary of a finitely generated fundamental group of a hyperbolic 3-manifold onto its limit set on the boundary in hyperbolic 3-space. The question whether a similar statement holds more generally occurs in Gromov (1993). In particular, that author asks whether every nonelementary finitely generated subgroup of a word hyperbolic group has nontrivial boundary.

After writing the first version of this paper I became aware of the pages 257–259 and 263–268 of Gromov's substantial essay (Gromov, 1993). The discussion there appears to overlap with the present paper

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and several of our results seem to occur already in this reference. The reader is of course encouraged to read the indicated pages in Gromov (1993) although there are some omissions and inadequacies. I hope that the arguments in the present paper may be of some additional interest as they avoid using compactness.

II. PRELIMINARIES

Let Γ be a group generated by a (finite or) countable set S. The group Γ can be viewed as a (metric) graph called the *Cayley graph* $C(\Gamma, S)$. The elements in Γ are the vertices and two vertices g, g' are connected by an edge if there is an $s \in S$ such that $g = g's^{\pm 1}$. Each edge is assigned length 1 (the edges are isometric to a unit interval). This defines lengths of paths in this graph and we can define a corresponding distance d. In this way $(C(\Gamma, S), d)$ is a complete geodesic space. The distance d is the *word metric*.

For a finitely generated group, this metric space is well-defined up to quasi-isometry, in other words if we change S to another finite generating set the two metric graphs will be quasi-isometric. (When S is allowed to be, or has to be infinite, pathologies may occur, for example: Γ is infinite but the graph has finite diameter.) Even so, we will suppress the dependence on S and simply denote the metric graph by Γ and its distance d and speak of geodesics (distance minimizing paths) as a subset of Γ in the obvious fashion. A geodesic path between $z, w \in \Gamma$ is usually denoted by $\gamma_{z,w}$. The distance from a point y to a subset A is

$$d(y, A) := \inf_{a \in A} d(y, a).$$

The action by Γ on itself (or $C(\Gamma, S)$) by left translation is an isometric action. For more details on these standard concepts see Bridson and Haefliger (1999).

We now wish to define a boundary of Γ following the construction in Floyd (1980) "which is based on an idea of Thurston's and inspired by a construction of Sullivan's", see also Gromov (1987). Let f be a function $\mathbb{N} \to \mathbb{R}_{>0}$ such that $\lambda f(r) \leq f(r+1) \leq f(r)$ for some constant $\lambda > 0$ and every r. The important property we require in addition for f is that of summability:

$$\sum_{j=0}^{\infty} f(r) < \infty.$$

(For example, we may consider $f(r) = 1/r^2$ for $r > 0$.)

We define a new distance d' by modifying the length of the edges. Let e denote the identity in the group, for two adjacent vertices g, h we define

$$f(d(e,\{g,h\}))$$

to be the length of an edge connecting them (instead of 1). This defines d'-length L_f of a path $\alpha = \{x_i\}$ in the graph:

$$L_f(\alpha) = \sum_i f(d(e,\{x_i, x_{i+1}\}))$$

and the new distance is

$$d'(z,w) := \inf_{\alpha} L_f(\alpha),$$

where the infimum is taken over all paths α connecting z and w. It is straightforward to verify that d' satisfies the axioms of a metric. In particular, since (Γ, d) is a geodesic space any two points z, w can be joined by a geodesic β, so $d'(z, w) \leq L_f(\beta) < \infty$. (When we speak about geodesics it will always refer to the distance d.) Note also that Γ has finite d'-diameter because f is summable.

We now define $\bar{\Gamma}$ to be the completion of (Γ, d') in the sense of metric spaces and the boundary is $\partial\Gamma = \bar{\Gamma}\backslash\Gamma$. The metric structure d' gives rise to a topology on this completion and boundary. Note that we are suppressing the dependence of S and f, and refer to $\partial\Gamma$ obtained as above as a *Floyd boundary of* Γ.

As mentioned above Γ acts on its Cayley graph by isometries. This action extends to an action of Γ by homeomorphisms of $\bar{\Gamma}$. To see this, note that for $s \in S$ we have

$$|d(e,sw) - d(e,w)| \leq d(e,s) = 1$$

for any $w \in \Gamma$. Together with the basic assumptions on f we immediately have that

$$\lambda d'(z,w) \leq d'(sz,sw) \leq \lambda^{-1} d'(z,w).$$

This estimate shows that $s^{\pm 1}$ takes Cauchy sequences to Cauchy sequences, equivalent ones to equivalent ones, in a continuous fashion. Finally, since s is an isometry, the map $s : \bar{\Gamma} \to \bar{\Gamma}$ (and $\partial\Gamma \to \partial\Gamma$) is a bijection. Hence the elements in S and therefore $\langle S \rangle = \Gamma$ acts by homeomorphisms of $\bar{\Gamma}$.

III. CONTRACTIVE PROPERTIES

Let Γ, S, d and f be as in the previous section. The following lemma is crucial for the present paper.

Lemma 1. *There is a function $G:\mathbb{N}\to\mathbb{R}_{>0}$ (depending only on f) such that $G(r)\to 0$ whenever $r\to\infty$ and having the following property. For any two points z and w in Γ and $\gamma_{z,w}$ a d-geodesic segment connecting z and w, it holds*

$$d'(z,w) \le G(d(e,\gamma_{z,w})).$$

Proof. Let a denote the distance to z from a point m on $\gamma_{z,w}$ closest to e and $r := d(e, m) = d(e,\gamma_{z,w})$. Let x_j, $j = 0,\dots, a$ be the points (vertices) of the geodesic segment $\gamma_{z,m}\subset\gamma_{z,w}$, where $x_0 = m$ and $x_a = z$. Because of the minimality of r and the triangle inequality we have the following estimates:

$$d(e,x_j) \ge r$$
$$d(e,x_j) \ge j - r.$$

We hence get, using monotonicity and summability of f:

$$\begin{aligned} d'(m,z) &\le \sum_{j=0}^{a-1} f(d(e,\{x_j,x_{j+1}\})) \\ &\le \sum_{j=0}^{2r-1} f(r) + \sum_{j=2r}^{a-1} f(j-r) \\ &\le 2rf(r) + \sum_{j=r}^{\infty} f(j) =: \frac{1}{2}G(r). \end{aligned}$$

(If $a < 2r-1$, then we would not decompose the sum, and so the second term would not be present.) By the same consideration with w instead of z and the triangle inequality, it only remains to prove that $G(r)\to 0$ whenever $r\to\infty$. But this is a simple consequence of the monotonicity and the summability of f. Indeed, given $\varepsilon > 0$ and using $f(j+1) < f(j)$ we see that

$$rf(r) \le 2(f(\lfloor r/2\rfloor - 1) + f(\lfloor r/2\rfloor) + \cdot + f(r)) \le 2\sum_{j=\lfloor r/2\rfloor}^{\infty} f(j) < \varepsilon$$

for large enough r because of the summability of f. □

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Let

$$(z|w) = \frac{1}{2}(d(e,z) + d(e,w) - d(z,w))$$

be the so-called Gromov product. It is a simple fact (see e.g. Karlsson and Noskov, 2002a) that

$$(z|w) \leq d(e,[z,w])$$

for any geodesic segment $[z, w]$. Following a nice argument of Woess in (1993) dealing with Gromov hyperbolic, proper metric spaces we can now prove the following *contraction property*:

Proposition 3. *Let g_n be a sequence in Γ. If $g_n \to \xi \in \partial\Gamma$ and $g_n^{-1} \to \eta \in \partial\Gamma$, then $g_n z \to \xi$ for any $z \in \bar{\Gamma}\backslash\{\eta\}$ and this convergence is uniform outside any neighborhood of η.*

Proof. (Note that for any $z \in \Gamma$, we clearly have that $g_n z \to \xi$, because $d(g_n, g_n z) = d(e, z)$.) Let U and V be neighborhoods in $\bar{\Gamma}$ of ξ and η respectively. By definitions, we can find a small $\varepsilon > 0$ and large $n_0 > 0$ such that

$$\{g_n : n \geq n_0\} \cup \{\xi\} \subset B_{\varepsilon/3}(g_{n_0}) \subset B_{\varepsilon}(g_{n_0}) \subset U$$

and

$$\{g_n^{-1} : n \geq n_0\} \cup \{\eta\} \subset B_{\varepsilon/3}(g_{n_0}^{-1}) \subset B_{\varepsilon}(g_{n_0}^{-1}) \subset V.$$

Here $B_a(x)$ denotes the open metric ball with center x and radius a in distance d'. Let $z \in \Gamma$ with $d'(g_{n_0}^{-1}, z) \geq 2\varepsilon/3$, so $d'(z, g_n^{-1}) \geq \varepsilon/3$ for any $n \geq n_0$. In view of Lemma 1 there is for any $\delta > 0$ a number $R(\delta) := \max G^{-1}([\delta, \infty))$ such that $G(r) < \delta$ for every $r \geq R(\delta)$. We therefore have for any $n \geq n_0$ that

$$d(e,[z,g_n^{-1}]) < R(\varepsilon/3).$$

But then

$$\begin{aligned} d(e,[g_n,g_n z]) &\geq (g_n|g_n z) \\ &= \frac{1}{2}(d(e,g_n) + d(e,g_n z) - d(z,e)) \\ &= d(e,g_n) - \frac{1}{2}(d(e,g_n) - d(g_n^{-1},z) + d(z,e)) \\ &= d(e,g_n) - (g_n^{-1}|z) \\ &\geq d(e,g_n) - d(e,[g_n^{-1},z]) \\ &\geq d(e,g_n) - R(\varepsilon/3), \end{aligned}$$

which tends to infinity as $n \to \infty$. Again by Lemma 1 there is therefore $n_1 \geq n_0$ such that

$$d'(g_n z, g_n) < \varepsilon/3,$$

for every z as above and $n \geq n_1$. As every $\zeta \in \partial\Gamma \setminus V$ can be approximated by elements z in Γ outside of $B_{2\varepsilon/3}(g_{n_0}^{-1})$ and by continuity of the action of Γ, the proposition is proved. □

In view of the last two statements, when Γ is finitely generated and $\partial\Gamma$ nontrivial, Γ acts as a convergence group (Gehring and Martin, 1987) on its boundary. In fact, Lemma 1 shows that Γ acts as a geometric convergence group in the sense of Gerasimov (2002).

Inspired by an argument on p. 264 of Gromov (1993) we prove the following lemma, which we will use at one point in the next section in combination with Proposition 3.

Lemma 2. *Let g_n be an unbounded sequence in Γ, that is, $d(g_n, e)$ is unbounded. Assume there is a point $\xi \in \partial\Gamma$ such that $g_n\xi \to \xi$. Then there is a subsequence n_k such that at least one of g_{n_k} or $g_{n_k}^{-1}$ converges to ξ.*

Proof. Let x_j be a d'-Cauchy-sequence representing ξ, so $x_j \to \xi$. Assume first that for any $R > 0$ there exists an N such that

$$(g_n | g_n x_j) \geq R$$

for all $n \geq N$ and $j \geq 1$. Then for any $\varepsilon > 0$ it holds for every sufficiently large n and j that

$$d'(g_n, \xi) \leq d'(g_n, g_n\xi) + \varepsilon \leq d'(g_n, g_n x_j) + 2\varepsilon \leq 3\varepsilon$$

in view of Lemma 1. This proves the lemma in this case.

In the complementary case we hence have that there is an $R > 0$ and subsequences $n_k \to \infty$ and j_k such that

$$(g_{n_k} | g_{n_k} x_{j_k}) < R.$$

As in the proof of Proposition 3 we have

$$(g_{n_k} | g_{n_k} x_{j_k}) = d(e, g_{n_k}) - (g_{n_k}^{-1} | x_{j_k}).$$

As $g_{n_k} \to \infty$, we have that $(g_{n_k}^{-1} \mid x_{j_k}) \to \infty$ and so in particular x_{j_k} has to go to infinity, hence converging to ξ. Lemma 1 now implies that $g_{n_k}^{-1}$ also converges to ξ as desired. □

IV. NONELEMENTARY SUBGROUPS

We now turn to the question of the existence of *doubly convergent sequences* $\{g_n\}_0^\infty \subset \Gamma$, that is, $g_n \to \xi^+$ and $g_n^{-1} \to \xi^-$ for some boundary points ξ^+ and ξ^-. We call ξ^+, ξ^- the *limit points* of the sequence (it is allowed that $\xi^+ = \xi^-$). An element $g \in \Gamma$ is called *unbounded* if $d(e, g^n)$ is unbounded in n. A subgroup Λ is called *nonelementary* with respect to $\partial\Gamma$ if it contains a doubly convergent sequence such that Λ does not stabilize its limit point set. The *limit set* of a subgroup Λ consists of every element $\xi \in \partial\Gamma$ which can be represented by a d'-Cauchy sequence with elements only in Λ.

In the case of infinite number of generators the existence of doubly convergent sequences does not seem so clear. Here is one argument which may apply to a nontorsion group:

Proposition 4. *Assume that $g \in \Gamma$ is unbounded. Then there exists a subsequence n_i such that $h_i := g^{n_i}$ is a doubly convergent sequence. The limit point(s) $\xi^\pm$ are the unique fixed points of g. In the case $d(e, g^n) \to \infty$, then both g^n and g^{-n} converge to points (or the same point) in $\partial\Gamma$.*

Proof. This follows an argument in Karlsson (2001). Let $a_n = d(e, g^n)$. Select $n_i \to \infty$ such that

$$a_{n_i} > a_m$$

for every $m < n_i$. Then for every $k < n_i$

$$\begin{aligned}(g^{n_i}|g^k) &= \frac{1}{2}(d(e, g^{n_i}) + d(e, g^k) - d(g^{n_i}, g^k)) \\ &\geq \frac{1}{2}a_k.\end{aligned}$$

In view of Lemma 1 and since $a_{n_i} \to \infty$, the sequence $h_i := g_{n_i}$ is a Cauchy sequence and hence converges to a point in $\partial\Gamma$. For the same reason, since $d(e, g^{-n}) = d(e, g^n)$, also the sequence h_i^{-1} converges in $\bar{\Gamma}$ to a point in $\partial\Gamma$. By continuity and contractivity we have

$$g(\xi^\pm) = g(\lim_{i\to\infty} g^{\pm n_i}) = \lim_{i\to\infty} g^{\pm n_i} g = \xi^\pm$$

and that no other point can be fixed. □

Note that in the case S is finite, an element g is unbounded if and only if $d(e, g^n) \to \infty$, because otherwise $g^N = 1$ for some $N \geq 1$.

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Lemma 3. *Assume $g \in \Gamma$ such that*

$$g(\partial\Gamma \setminus U^-) \subset U^+$$

for two disjoint nonempty sets U^+ and U^-. Then g generates an infinite cyclic group. If each d-metric ball contains at most finitely many distinct elements of the form g^k, then $g^k \to \infty$.

Proof. It is clear that $g^k(\partial\Gamma \setminus U^-) \subset U^+$ for $k > 0$, since $U^+ \subset \partial\Gamma \setminus U^-$, and hence g^k cannot be the identity. (If $U^- = U^+$ then g could have order 2.) If some subsequence g^{n_k} stays inside of a ball which have only finitely many distinct elements of this form, then $g^{n_k} = g^{n_l}$ for two distinct indices, which implies $g^k = 1$ for a nontrivial k. □

By adapting a nice argument of Gehring and Martin (1987), see Karlsson and Noskov (2002b), we get:

Proposition 5. *Let g and h be two unbounded elements in Γ. Assume that every infinite subset of the subgroup generated by g and h contains an unbounded sequence. Then the fixed point sets of g and h are either equal or disjoint.*

Proof. Both fixed point sets are nonempty by Proposition 4. In the case both g and h have only one fixed point, the statement is trivial. Assume now that $Fix(h) = \{\xi^+, \xi^-\}$, $(h^{\pm n_j} \to \xi^{\pm})$ and $\xi^- \in Fix(g)$. We need to show that ξ^+ is fixed also by g and we may therefore assume that $\partial\Gamma$ contains at least three (hence infinite number of) points. Choose neighborhoods U^-, U^+ in $\bar{\Gamma}$ of ξ^- and ξ^+ respectively so that

$$hU_- \cap U_+ = \emptyset, \tag{IV.1}$$

which is possible because h is continuous and $\xi^{\pm}$ are fixed points of h. Let $E = \bar{\Gamma} \setminus (U^+ \cup U^-) \neq \emptyset$. Since negative powers of h contracts toward ξ^-, g is continuous and fixes ξ^-, we have that

$$gh^{-n_j}(E) \subset U_- \setminus \{\xi^-\}$$

for every large j. Because of (IV.1) we may pick the smallest $k = k(j)$ such that

$$h^{k(j)} gh^{-n_j} E \cap E \neq \emptyset. \tag{IV.2}$$

Now let $g_j = h^{k(j)} gh^{-n_j}$ and note that

$$g_j \xi^- = \xi^- \quad \text{and} \quad \lim_{j \to \infty} g_j \xi^+ = \xi^+, \tag{IV.3}$$

since $k(j) \to \infty$ as $j \to \infty$ and $gh^{-n_j}\xi^+ = g\xi^+$.

We assert that g_j is bounded. Suppose not, then by Lemma 2 (or by compactness if S is finite) there is a subsequence n_i such that $g_{n_i}^{\pm 1} x \to \eta^{\pm} \in \partial\Gamma$. From (IV.3) it is clear that $\{\eta^{\pm}\} = \{\xi^{\pm}\}$. But from the contraction property we should also have that $g_{n_i} E \subset U^+$ for all large i, which contradicts (IV.2). Hence g_j is bounded and by the properness assumption on Λ, $g_j = g_i$ for some distinct i and j. Therefore $h^k = gh^l g^{-1}$ for two nonzero integers k and l.

We claim that it now follows that $g\xi^+ = \xi^+$. Applying the obtained identity to $g\xi^+$ we have

$$h^k(g\xi^+) = gh^l g^{-1} g\xi^+ = g\xi^+.$$

As $\bar{h} := h^k$ is unbounded it can have at most two fixed points, namely the same as h, that is, $\xi^{\pm}$. So we have that $g\xi^+$ is either ξ^+ or ξ^-, but the latter is impossible because $g\xi^- = \xi^-$ and g is bijective. □

Proposition 6. *Assume that Γ is generated by a finite set S and let $\partial\Gamma$ be a (nontrivial) Floyd boundary. If the limit set in $\partial\Gamma$ of a subgroup Λ contains at least three points, then it is nonelementary with respect to $\partial\Gamma$. In particular, if $\partial\Gamma$ is nontrivial, then Γ is nonelementary.*

Proof. The existence of doubly convergent sequences is a simple consequence of compactness: given $\xi \in \partial\Gamma$ in the limit set, take any Cauchy sequence g_n of points in Λ converging to ξ, and select by sequential compactness a subsequence n_k such that also $g_{n_k}^{-1}$ converges, say to ξ^-. Let h_n be another doubly convergent sequence in Λ but with forward limit point η different from ξ and ξ^-. If it is the case that h_n^{-1} converges to a point different from ξ^+ or ξ^+ then we are done by Proposition 3. Therefore we may now assume that $\xi = \xi^-$.

As $g_n\eta \to \xi$ by the contraction property, it follows that either the orbit $\{g_n\eta\}_{n>0}$ contains infinite number of points or $g_n\eta = \xi$ for all large n. In the former case, the proposition is proved. In the latter case, as all elements are bijections, $g_n\xi \neq \xi$ for all large n. Hence, in any case we have found a doubly convergent sequence whose limit points are not invariant as a set under Λ. □

V. NONCOMMUTATIVE FREE SUBGROUPS

We now prove Theorem 1. Let g_n be a doubly convergent sequence with limit points ξ^+ and ξ^-. Assume that $p \in \Lambda$ is such that $p\xi^+ \notin \{\xi^+, \xi^-\}$. Note that by Proposition 3, g_n contracts all of $\partial\Gamma \setminus \{\xi^-\}$

towards ξ^+, and that $pg_n z \to p\xi^+$ for z outside ξ^- and $(pg_n)^{-1} z = g_n^{-1} p^{-1} z \to \xi^-$ for z outside $p\xi^+$.

In view of Proposition 3, Lemma 3, and $p\xi^+ \notin \{\xi^+, \xi^-\}$, we can find N such that $g^k \to \infty$ for $g := pg_N$ (by the properness assumption) and the two limit points ξ_+ and ξ_- (which exist by Proposition 4) are distinct.

Since the point $p\xi^+$ is contracted by g_n towards ξ^+, the set $\{g_n p\xi^+ : n > 0\} \cup \{g_n \xi^+ : n > 0\}$ is infinite: the sequences $g_n p\xi^+$ and $g_n \xi^+$ must get arbitrarily close to each other (and close to ξ^+) without ever coincide (because g_n is invertible). Thus we can find η of the form $g_M p\xi^+$ or $g_M \xi^+$, different from ξ^+, ξ^-, ξ_+, ξ_- and $p\xi^+$, and such that for $h_n := g_M p g_n$ or $h := g_M g_n$ we have $h_n \to \eta$ and $h_n^{-1} \to \xi^-$.

Therefore we can again (as with pg_n and g above) find a number L such that $h := h_L$ is an unbounded element with two distinct fixed points $\eta_\pm$ such that η_+ is different from ξ_+ and ξ_-. Proposition 5 implies that also $\eta_- \notin \{\xi_-, \xi_+\}$.

Since all the four limit points $\xi_\pm$, $\eta_\pm$ are different and in view of Proposition 3, we may now use the standard so-called ping-pong lemma (see e.g., de la Harpe, 2000 or Tits, 1972) on some powers of g and h to conclude the proof of the theorem.

VI. STRONGLY PROXIMAL BOUNDARIES

In Furstenberg (1973) defined a boundary of a group which records contractive phenomena and which are opposite to amenability. A compact metrizable space X on which Γ acts by homeomorphisms is called a *boundary in the sense of Furstenberg* if it is minimal, meaning that every Γ-orbit is dense, and strongly proximal, meaning that $\overline{\Gamma\mu}$ contains point-measures for every probability measure μ on X.

We now give the proof of Theorem 2. Assume that Γ is generated by a finite set S and $\partial\Gamma$ is nontrivial.

From Sec. II we know that $\partial\Gamma$ is a compact, metrizable space on which Γ acts by homeomorphisms.

Assume that a nonempty closed subset $A \subset \partial\Gamma$ is Γ-invariant. By the contraction property, every doubly convergent sequence must have at least one limit point in A. The only possibility (in view of the existence of doubly convergent sequences) is that $A = \{\xi\}$ or $A = \partial\Gamma$. If A is just one point ξ, then because of the nontriviality of $\partial\Gamma$ we can find two doubly convergent sequences $h_n \to \eta_1$ and $f_n \to \eta_2$ with η_1, η_2 and ξ distinct. As in Sec. V we can now find h and f such that h^k converges to η_{11} and f^k converges to η_{22}, again distinct and different from ξ. By the invariance of A we have that $h^{-k} \to \xi$ and $f^{-k} \to \xi$, but this contradicts Proposition 5.

As in Sec. V we know that $\partial\Gamma$ is infinite. Moreover, for any $\xi \in \partial\Gamma$ we can find a sequence of group elements $g_n \to \xi$ and we can assume that this sequence is doubly convergent thanks to sequential compactness. It follows that $\partial\Gamma$ is a perfect set, hence uncountable. Finally, as a probability measure μ cannot have uncountably many atoms we see that we can pick a nonatom $\xi \in \partial\Gamma$ and a doubly convergent sequence g_n such that $g_n^{-1} \to \xi$. Proposition 3 now guarantees the required strong proximality property.

VII. SOME FURTHER STATEMENTS

In analogy with Hopf's result we have:

Proposition 7. *The cardinality of* $\partial\Gamma$ *is either* 0, 1, 2, *or uncountable. The group* Γ *is virtually* $\mathbb{Z}$ *if and only if* $|\partial\Gamma| = 2$.

Given Proposition 3, the proof is standard, see Gehring and Martin (1987, Theorem 4.5), Woess (1993, Theorem 2(ii)), and Bridson and Haefliger (1999, I.8.32). The observation that $|\partial\Gamma| \geq 3$ implies $|\partial\Gamma| = \infty$ can essentially be found above.

Proposition 8. *If* Γ, *a group generated by a finite set S,*

- *does not contain a free nonabelian subgroup, or more generally,*
- *has an infinite normal subgroup not containing a free nonabelian subgroup, or,*
- *is a direct product of two infinite groups,*

then any Floyd boundary contains at most two points.

Proof. The first point is an immediate consequence of the statements in the introduction. Now assume Λ is an infinite normal subgroup not containing F_2. Then the limit set $L(\Lambda)$ must consist of either one or two points. On the other hand as Λ is a normal subgroup,

$$L(\Lambda) = L(g\Lambda g^{-1}) = L(g\Lambda) = gL(\Lambda),$$

for any g which shows that the limit set is Γ invariant, hence $\partial\Gamma = L(\Lambda)$. The last criterium for triviality was already observed by Floyd, but here is a dynamical proof: Assume $\Gamma = \Lambda_1 \times \Lambda_2$. Then $L(\Lambda_1 \times 1)$ is nonempty and Λ_2 fixes this set pointwise. As also Λ_2 is infinite, the statement follows. □

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Finally we record the following which is implicitly already proved above.

Proposition 9. *Let Γ be a group generated by a set S and assume that a Floyd boundary $\partial\Gamma$ is nontrivial. If Γ contains an unbounded element, then it contains a free nonabelian semigroup.*

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