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# Hahn-Banach for metric functionals and horofunctions

Anders Karlsson [a,b,*,1]

[a] *Section de mathématiques, Université de Genève, 2-4 Rue du Lièvre, Case Postale 64, 1211 Genève 4, Switzerland*
[b] *Matematiska institutionen, Uppsala universitet, Box 256, 751 05 Uppsala, Sweden*

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A B S T R A C T

It is observed that a natural analog of the Hahn-Banach theorem is valid for metric functionals but fails for horofunctions. Several statements of the existence of invariant metric functionals for individual isometries and 1-Lipschitz maps are proved. Various other definitions, examples and facts are pointed out related to this topic. In particular it is shown that the metric (horofunction) boundary of every infinite Cayley graphs contains at least two points.


## 1. Introduction

It is well-recognized that the Hahn-Banach theorem concerning extensions of continuous linear functionals is a cornerstone of functional analysis. Its origins can be traced to so-called moment problems, and in addition to H. Hahn and Banach, Helly's name should be mentioned in this context ([12,40]).

* Correspondence to: Section de mathématiques, Université de Genève, 2-4 Rue du Lièvre, Case Postale 64, 1211 Genève 4, Switzerland.
*E-mail addresses:* anders.karlsson@unige.ch, anders.karlsson@math.uu.se.
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In an important recent development, called the *Ribe program*, notions from geometric Banach space theory are formulated purely in terms of the metric associated with the norm, and then studied for metric spaces. This has been developed by Bourgain, Ball, Naor and others, for a recent partial survey, see [38]. This subject is in part motivated by significant applications in computer science.

It has been remarked in a few places (see [47,22] for merely two references, the earliest discussion would surely be found in Busemann's work in the 1930s, see [39] for a good exposition of this) that Busemann functions or horofunctions serve as replacement for linear functions when the space is not linear. This has been a useful notion in the theory of manifolds with non-negative curvature ([10,6,14]) as well as for spaces of non-positive curvature ([3,7,5]), and more recently several other contexts beyond any curvature restriction, for example [24,33,31,32,45,46]. In parallel, they have been identified or described for more and more metric spaces, see [27] for references.

In the present paper, we continue to consider the analogy in this direction between the linear theory and metric theory. More precisely, we consider the category of metric spaces and semi-contractions (non-expanding maps or 1-Lipschitz maps), that is, maps $f : X \to Y$ which do not increase distances:

$$d(f(x_1), f(x_2)) \leq d(x_1, x_2)$$

for all $x_1, x_2 \in X$. As discussed below one has metric analogs of the norm, spectral radius, weak topology, the Banach-Alaoglu theorem, and the spectral theorem. The present paper focuses on pointing out an analog of the Hahn-Banach theorem. To our mind, the circumstance that fruitful such analogs are present is surprising and promising.

We follow Banach in his paper [4] and in his classic text *The Theory of Linear Operators* from 1932 in calling maps from a metric space $X$ into $\mathbb{R}$ *functionals*. Since we in addition consider semi-contractions as our morphisms, we prefix the functionals that we will consider by the word *metric*. These *metric functionals* generalize Busemann functions and horofunctions, see the next section for precise definitions of the latter two concepts, and are moreover an analog and replacement for continuous linear functionals in standard functional analysis, perhaps further motivating the use of the word *functional*.

To be precise, with a fixed origin $x_0 \in X$, the following functions and their limits in the topology of pointwise convergence are called *metric functionals*:

$$h_x(\cdot) := d(\cdot, x) - d(x_0, x).$$

The closure of this continuous injection of $X$ into functionals, is called the *metric compactification* of $X$ and is compact by the analog of Banach-Alaoglu, see below. If $X$ is proper and geodesics we call the subspace of new points obtained with the closure the *metric boundary*. In discussions with Tobias Hartnick, we observed the following statement (and as will be explained the corresponding statement for horofunctions is false).

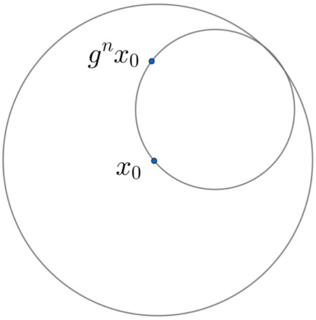


**Fig. 1.** A classical parabolic isometry.

**Proposition 1.** *(Metric Hahn-Banach statement) Let* $(X, d)$ *be a metric space with base point* $x_0$ *and* $Y$ *a subset containing* $x_0$*. Then for every metric functional* $h$ *of* $Y$ *there is a metric functional* $H$ *of* $X$ *which extends* $h$ *in the sense that* $H|_Y = h$*.*

Because the above is a direct consequence of compactness, it is a bit artificial to speak of applications of it in a strict sense. However, it does provide a point of view that we believe is useful.

Note that general Cayley graphs are highly non-trivial, so the following statement is not an obvious fact:

**Theorem 2.** *Let* $\Gamma$ *be a finitely generated, infinite group and* $X$ *a Cayley graph defined by* $\Gamma$ *and a finite set of generators. Then the metric boundary of* $X$ *contains at least two points, and each metric functional is unbounded.*

Fixed point theorems are of fundamental importance in analysis. The following results are related to fixed point statements:

**Theorem 3.** *For any monotone distorted isometry* $g$ *of a metric space* $X$*, there exists a metric functional that vanishes on the whole orbit* $\{g^n x_0 : n \in \mathbb{Z}\}$*.*

Thus this result shows that the classical picture of parabolic isometries preserving horospheres in hyperbolic geometry extends, see Fig. 1. These two theorems are proved in section 5 where also some further results about metric functionals of general Cayley graphs are discussed.

With Bas Lemmens we observed the following improvement of the theorem but under another hypothesis on the map:

**Proposition 4.** *Suppose that* $f$ *is a semi-contraction of a metric space* $X$ *with* $\inf_x d(x, f(x)) = 0$*. Then there is a metric functional* $h$*, such that*

$$h(f(x)) \leq h(x)$$

*for all $x \in X$. In case $f$ is an isometry the inequality is an equality.*

It is an improvement in the sense that if $f$ is an isometry (with the alternative assumptions) then $h(f^n x_0) = h(x_0) = 0$ for all $n$.

Bader and Finkelshtein defined a *reduced boundary* by identifying functions in the metric boundary if they differ by a bounded function [2]. We note that:

**Proposition 5.** *Every isometry of a metric space fixes a point in the reduced metric compactification.*

Busemann wrote in 1955 that "... two startling facts: much of Riemannian geometry is not truly Riemannian and much of differential geometry requires no derivatives". Although it would be too much to affirm that much of functional analysis requires no linear structure, at least there are a number of analogs for general metric spaces.

**Acknowledgments:** It is a pleasure to thank Tobias Hartnick for the invitation to Giessen and our discussion there that sparked the origin of this note. I also thank Nate Fisher, Pierre de la Harpe, and Bas Lemmens for useful discussions.

## 2. Basic definitions and terminology

Let $(X, d)$ be a metric space. One defines (a variant of the map considered by Kuratowski and Kunugui in the 1930s, see for example [29, p. 45])

$$\Phi : X \to \mathbb{R}^X$$

via

$$x \mapsto h_x(\cdot) := d(\cdot, x) - d(x_0, x).$$

As the notation indicates the topology we take here in the target space is pointwise convergence. The map is continuous and injective.

**Proposition 6.** *(Metric Banach-Alaoglu) The space $\overline{\Phi(X)}$ is a compact Hausdorff space.*

**Proof.** By the triangle inequality we note that

$$-d(x_0, y) \leq h_x(y) \leq d(y, x_0),$$

which implies that the closure $\overline{\Phi(X)}$ is compact by the Tychonoff theorem. It is Hausdorff, since it is a subspace of a product space of Hausdorff spaces, indeed metric spaces. □

In general this is not a compactification of $X$ in the strict and standard sense that the space is embedded, but it is convenient to still call it a compactification. In other words, we provide a weak topology that has compactness properties.

By the triangle inequality it follows that all the elements in $\overline{X} := \overline{\Phi(X)}$ are semi-contractive functionals $X \to \mathbb{R}$, and we call them *metric functionals*.

**Example 7.** Let $\gamma$ be a geodesic line (or just a ray $\gamma : \mathbb{R}_+ \to X$), which is a standard notion in metric geometry at least since Menger [35]. Then the following limit exists:

$$h_\gamma(y) = \lim_{t\to\infty} d(y, \gamma(t)) - d(\gamma(0), \gamma(t)).$$

The reason for the existence of the limit for each $y$ is that the sequence in question is bounded from below and monotonically decreasing (thanks to the triangle inequality), see [3,5]. This element in $\overline{\Phi(X)}$ is called the *Busemann function associated with* $\gamma$.

**Example 8.** The open unit disk of the complex plane admits the Poincaré metric, which in its infinitesimal form is given by

$$ds = \frac{2\,|dz|}{1 - |z|^2}.$$

This gives a model for the hyperbolic plane and moreover every holomorphic self-map of the disk is a semi-contraction in this metric (the Schwarz-Pick lemma). The Busemann function associated to the (geodesic) ray from 0 to the boundary point $\zeta$, in other words $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, is

$$h_\zeta(z) = \log \frac{|\zeta - z|^2}{1 - |z|^2}.$$

These functions appear (in disguise) already in 19th century mathematics, such as in the Poisson integral representation formula and in Eisenstein series.

The more common choice of topology, introduced by Gromov in [17], is uniform convergence on bounded sets. We denote the corresponding closure $\overline{X}^h$ and call it the *horofunction bordification*, the new points are called *horofunctions* following common terminology. This closure amounts to the same compactification if $X$ is proper (i.e. closed bounded sets are compact), see [5], but in general it is quite different, in particular there is no notion of weak compactness. As in the example described in Remark 13 below, a space may have no horofunctions. Moreover, the following example shows that Busemann functions are not always horofunctions, since the limit above might not converge in this topology:

**Example 9.** Take one ray $[0, \infty]$ that will be geodesic $\gamma$, then add an infinite number of points at distance 1 to the point $x_0 = 0$ and distance 2 to each other. Then

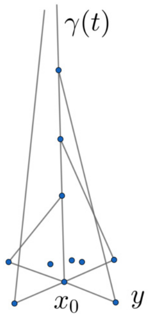


**Fig. 2.** The ray $\gamma$ does not define a horofunction.

at each point $n$ on the ray, connect it to one of the points around 0 (that has not already been connected) with a geodesic segment of length $n - 1/2$. This way $h_\gamma(y) = \lim_{t\to\infty} d(\gamma(t), y) - d(\gamma(t), \gamma(0))$ still of course converge for each $y$ but not uniformly. Hence the Busemann function $h_\gamma$ is a metric functional but not a horofunction. See Fig. 2.

The topology that we chose here has been useful in a few instances already: [32,13, 16,15,34] and in on-going work by Bader and Furman (personal communication).

In a Banach space the metric functionals associated to points are not linear since they are closely related to the norm, but sometimes their limits are linear, see for example [27]. In any case they are all convex functions:

**Proposition 10.** *Let $E$ be a normed vector space. Every function $h \in \overline{E}$ is convex, that is, for any $x, y \in X$ one has*

$$h(\frac{x+y}{2}) \leq \frac{1}{2}h(x) + \frac{1}{2}h(y).$$

**Proof.** Note that for $z \in E$ one has

$$\begin{aligned} h_z((x+y)/2) &= \|(x+y)/2 - z\| - \|z\| = \frac{1}{2}\|x - z + y - z\| - \|z\| \\ &\leq \frac{1}{2}\|x - z\| + \frac{1}{2}\|y - z\| - \|z\| = \frac{1}{2}h_z(x) + \frac{1}{2}h_z(y). \end{aligned}$$

This inequality passes to any limit point of such $h_z$. $\square$

In [16, Lemma 3.1] it is shown that for any metric functional $h$ of a real Banach space there is a linear functional $f$ of norm at most 1 such that $f \leq h$. The proof uses the standard Banach-Alaoglu and Hahn-Banach theorems.

**Example 11.** Gutiérrez has provided a good description of metric functionals for $L^p$ spaces $p \geq 1$. To give an idea we recall the formulas for $\ell^p(J)$ for $1 < p < \infty$. There are two types:

$$h_{z,c}(x) = \left( \|x - z\|_p^p + c^p - \|z\|_p^p \right)^{1/p} - c,$$

where $z \in \ell^p(J)$ and $c \geq \|z\|_p$, as well as

$$h_\mu(x) = -\sum_{j \in J} \mu(j) x(j),$$

where $\mu \in \ell^q(J)$, with $q$ the conjugate exponent, and $\|\mu\|_q \leq 1$. See [18–20] for more details and precise statements. An interesting detail that Gutiérrez showed is that the function identically equal to zero is not a metric functional for $\ell^1$ (in contrast to the space $\ell^2$). He also observed how a famous fixed point free example of Alspach fixes a metric functional. For another discussion about Busemann functions of certain normed spaces, see [43,45].

## 3. Hahn-Banach theorem for metric functionals

**Proposition 12.** *(Metric Hahn-Banach statement.) Let $(X, d)$ be a metric space with base point $x_0$ and $Y$ a subset containing $x_0$. Then for every $h \in \overline{Y}$ there is a metric functional $H \in \overline{X}$ which extends $h$ in the sense that $H|_Y = h$.*

**Proof.** Given $h \in \overline{Y}$. Since the metric compactifications are Hausdorff (even metrizable if $X$ is separable) we take a net $h_{y_\alpha}$ that converges to the unique limit $h$. These points $y_\alpha$ are also points in $X$ and by compactness of $\overline{X}$ also has a limit point $H$ there. By uniqueness of the limits it must coincide with $h$. □

**Remark 13.** On the other hand, for horofunctions, i.e. for $\overline{X}^h$, the Hahn-Banach theorem does *not* hold. Example 9 shows this, with $Y$ taken to be the (image of the) geodesic ray $\gamma$. Then $Y$ clearly has a metric functional $b$ that is a Busemann function, however in $X$ any sequence of points going to infinity (i.e. along $Y$) cannot converge in $\overline{X}^h$, but to extend $b$ we would need such. Another illustration of this phenomenon (a counter-example to the proof but not the statement) is the following: Consider longer and longer finite closed intervals $[0, n]$ all glued to a point $x_0$. See Fig. 3. This becomes a countable (metric) tree which is unbounded but contains no infinite geodesic ray. Denote by $x_n$ the other end points of each interval. The sequence $h_{x_n}$ do not converge uniformly on balls. On the other hand it does so in $Y$ being just the set $\{x_n\}_{n \geq 0}$.

Here is a remark in another direction:

**Remark 14.** From a more general point of view, one could expect the possibility of extensions of metric functionals. The real line $(\mathbb{R}, |\cdot|)$ as a metric space is an injective object in the category we consider, namely metric spaces and semi-contractions (i.e. 1-Lipschitz maps), in the following restricted sense: Given any metric space $A$ and a

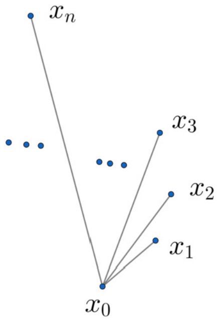


**Fig. 3.** A counter-example.

monomorphism that is isometric $\phi : A \to B$ where $B$ is another metric space, and semi-contraction $f : A \to \mathbb{R}$ there is an extension in the obvious sense of $f$ to a morphism $B \to \mathbb{R}$, for example

$$\bar{f}(b) := \sup_{a \in A} (f(a) - d(\phi(a), b))$$

or

$$\bar{f}(b) := \inf_{a \in A} (f(a) + d(\phi(a), b)) .$$

To see this, in the former definition, note first that for $b' = \phi(a')$ one has $f(a') - d(\phi(a'), \phi(a')) \geq f(a) - d(\phi(a), \phi(a'))$, which shows that our map is an extension, and then in general that

$$\begin{aligned}\overline{f}(b) - \overline{f}(b') &\leq \sup_a (f(a) - d(\phi(a), b)) - \sup_{a'} (f(a') - d(\phi(a'), b')) \\ &\leq \sup_a (f(a) - d(\phi(a), b) - f(a) + d(\phi(a), b')) \leq d(b, b'),\end{aligned}$$

which implies it is a morphism. The origin of this observation is [36]. Another possible reference in this context is [1].

There are topological vector spaces with trivial dual. Maybe in spirit this is a bit similar to the following example:

**Example 15.** Let $D : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be an increasing function with $D(0) = 0$, $D(t) \to \infty$ and $D(t)/t \to 0$ monotonically. The latter condition implies that $D(t+s) \leq D(t) + D(s)$, so one sees that $(\mathbb{R}, D(|\cdot|))$ is a metric space. As is observed in [26], $D(t)/t \to 0$ implies that

$$\overline{(\mathbb{R}, D(|\cdot|))} = \{h_x : x \in \mathbb{R}\} \cup \{h \equiv 0\} .$$

That is, there is a metric functional which vanishes on the whole space, and the compactification is the one-point compactification. Note that this metric space has no geodesics explaining in particular why there are no Busemann functions.

In contrast to the linear theory, note that not every metric functional of $X$ is a metric functional of a subset $Y$ as the following example illustrates:

**Example 16.** Let $X$ be the Euclidean space $\mathbb{R}^d$ and $Y = \mathbb{R}$ be a one dimensional linear subspace. The Busemann function associated to a ray from the origin perpendicular to $Y$ vanishes identically on $Y$. On the other hand the zero function is not a metric functional on $Y$.

## 4. Metric spectral notions

Let $f$ be a semi-contraction of a metric space to itself. As remarked in [27], the *minimal displacement* of $f$, $d(f) = \inf_x d(x, f(x))$, is the analog of the norm of a linear operator and the *translation number* is the analog of the spectral radius:

$$\tau(f) = \lim_{n\to\infty} \frac{1}{n} d(x, f^n(x)),$$

which exists in view of subadditivity. Similar to relationship between the norm and spectral radius, one always has that $\tau(f) \le d(f)$, since the translation number is independent of $x$. Note that in general the inequality may be strict, for example a rigid rotation of the circle, or more interestingly, for groups with a word metric all non-identity elements $g$ have $d(g) \ge 1$, but can easily have $\tau(g) = 0$.

In passing we record the following simple fact.

**Proposition 17.** *The following tracial property holds:*

$$\tau(gf) = \tau(fg)$$

*for any two semi-contractions $f$ and $g$.*

**Proof.** From the triangle inequality,

$$\begin{aligned} d(x, (fg)^n(x)) &\le d(x, f(x)) + d(f(x), (fg)^n(x)) \\ &\le d(x, f(x)) + d(x, (gf)^{n-1} g(x)) \\ &\le d(x, f(x)) + d(x, (gf)^n(x)) + d((gf)^n(x), (gf)^{n-1} g(x)) \\ &\le d(x, f(x)) + d(x, (gf)^n(x)) + d(f(x), x). \end{aligned}$$

Dividing by $n$ and sending $n$ to infinity shows one inequality. By symmetry also the opposite inequality holds. □

We say that an isometry $g$ is *distorted* if $\tau(g)=0$. Example 15 provides an observation related to distortion. This example could be considered for $\mathbb{Z}$ (instead of $\mathbb{R}$) and both are groups and the metric is invariant (but not a word-metric). Here all non-zero elements are distorted.

It was shown in [21], see also [27], that for any semi-contraction $f$ there is a metric functional $h$ such that

$$h(f^n(x_0)) \leq -\tau(f)n$$

for all $n>0$ and

$$\lim_{n\to\infty} -\frac{1}{n}h(f^n(x)) = \tau(f).$$

I will refer to this result as the *metric spectral principle*. As A. Valette pointed out to me, this could also be viewed as a statement in the spirit of the classical Hahn-Banach theorem: existence of a functional that realizes the norm of particular element.

In discussions with Bas Lemmens we observed the following, which strengthen the previous statement in a special case (similar, more detailed, strengthenings have been established in [13] and [32] in a variety of settings.):

**Proposition 18.** *Suppose that $f$ is a semi-contraction of $X$ with $d(f)=0$. Then there is a metric functional $h$, such that*

$$h(f(x)) \leq h(x)$$

*for all $x\in X$.*

**Proof.** For any $\epsilon>0$ we define the sets

$$N_\epsilon = \{x\in X : d(x,f(x))\leq\epsilon\}.$$

Take points $x_n \in N_{1/n}$ for $n>1$. Then by compactness $\{x_n\}$ has a subnet $\{x_\alpha\}$ converging to a metric functional $h$. So for each $n>1$ there exists an index $A_n$ such that $d(f(x_\alpha),x_\alpha)\leq 1/n$ for all $\alpha>A_n$. Hence for any $x\in X$

$$\begin{aligned} d(f(x),x_\alpha)-d(x_0,x_\alpha) &\leq d(f(x),f(x_\alpha))+d(f(x_\alpha),x_\alpha)-d(x_0,x_\alpha) \\ &\leq d(x,x_\alpha)-d(x_0,x_\alpha)+1/n, \end{aligned}$$

for all $\alpha\geq A_\alpha$. Taking the limit we obtain, since $n$ was arbitrary, that $h(f(x))\leq h(x)$ for all $x\in X$ as required. □

**Corollary 19.** *Suppose the displacement of an isometry $g$ is zero, then there is an invariant metric functional $h$, in the strong sense that $h(gx)=h(x)$ for all $x$.*

**Proof.** This is immediate from the previous proposition and its proof. □

Note that this is not in contradiction with the existence of complicated isometries like Edelstein's example, see [21], since $h \equiv 0$ is a metric functional for Hilbert spaces.

In the following section we will prove similar statements as above in the case when $\tau(g) = 0$ but $d(f)$ possibly strictly positive.

**The invariant subspace problem.** It is known that there are bounded operators of $\ell^1$ without non-trivial invariant closed subspaces, see [41]. Compare this with the following. For the purpose of invariant subspaces we can assume that we have an invertible operator $U : \ell^1 \to \ell^1$ with norm 1. Let $v \in \ell^1$ be a vector, and define the (affine) isometry $f(x) = Ux + v$. In [28] it is shown that there exists a continuous linear functional $l$ such that

$$\lim_{n\to\infty} l\left(\frac{1}{n}f^n(0)\right) = \lim_{n\to\infty} \frac{1}{n} \left\| f^n(0) \right\|.$$

The set of such linear functionals is invariant under $U$ and so if this limit is non-zero then the intersection of the kernels of these must be a non-trivial closed invariant subspace. So let us assume the limit is 0. In this case, by the corollary above, we have an invariant metric functional for $x \mapsto Ux + v$. This is a non-trivial function in view of Gutiérrez work ([19]) that shows that the zero function is not a metric functional. In this context, it is worth keeping in mind that for any non-zero metric functional $h$ there is a non-zero linear functional dominated by $h$ ([16]).

## 5. Group theory and parabolic isometries

Let me first record some facts that we, and probably others, have realized years ago, see [24,25]. They are however not generally known by people in geometric group theory. Let $X$ be a Cayley graph of a finitely generated group $\Gamma$, which becomes a metric space with the corresponding word metric associated to a finite generating set, see the recent book [11] for a wealth of metric geometry in this setting. For Cayley graphs we always take the neutral element as base point, $x_0 = e$. The group acts by isometry on $X$, and this action extends continuously to an action by homeomorphism of $\overline{X}$ and also on the *metric boundary* $\partial X := \overline{X} \setminus X$. This boundary is a compact metrizable space. Let $\lambda$ be a $\Gamma$-invariant probability measure on $\partial X$. Then the following map

$$T(g) := \int_{\partial X} h(g)d\lambda(h)$$

is a 1-Lipschitz group homomorphism $\Gamma \to \mathbb{R}$.

No metric functional on $X$ is identically zero, see below. Therefore if $\Gamma$ fixes a $h \in \partial X$ then there is a non-trivial homomorphism $T : \Gamma \to \mathbb{Z}$. A further idea shown in [25] is that given a finitely generated group with countable boundary, there is a finite index subgroup that surjects on $\mathbb{Z}$. One might wonder ([25]) or even conjecture ([42]) that every group of polynomial growth has countable boundary. This would immediately imply the celebrated theorem of Gromov that such groups are virtually nilpotent. Some positive evidence for this approach is provided in [44,42]. One could moreover have some hope that growth considerations in relation to the metric boundary could yield more than what is already known in this direction (recall that Grigorchuk's Gap Conjecture remains open).

It is well-known that infinite finitely generated groups (and their Cayley graphs) can be very complicated, for example they may contain only elements of finite order, or even an exponent $N$ such that $g^N = 1$ for all group elements $g$. Therefore I think that the following is a non-obvious fact.

**Theorem 20.** *Let $\Gamma$ be an infinite, finitely generated group and $X$ a Cayley graph associated to a finite set of generators. Then the metric boundary must contain at least two points and all metric functionals are unbounded, in the sense that for any metric functional $h$ there is no upper bound of $|h(x)|$ as a function of $x$.*

**Proof.** First we establish the second assertion. Note that given the assumptions, the graph $X$ is a proper metric space that is geodesic, in particular connected, and with infinite diameter. In view of the latter property, any $h_g$ with $g \in \Gamma$ is clearly unbounded. It remains to consider a limit function $h(x) = \lim_{n\to\infty} d(x, g_n) - d(e, g_n)$. Since the distance function is integer valued and the graph locally finite, it holds that for any $r > 0$ there is a number $N$ such that $h(x) = d(x, g_n) - d(e, g_n)$ for all $n \geq N$ and $x$ such that $d(x, e) \leq r$. Take a geodesic from $e$ to $g_N$, it must intersect the sphere around $e$ of radius $r$ in a point $y$. This means that $h(y) = d(y, g_N) - d(g_N, e) = -r$. The statement follows since $r$ was arbitrary.

Now assume that the metric boundary contains exactly one point, $\partial X = \{h\}$. By the remarks above, $h$ defines a homomorphism $\Gamma \to \mathbb{Z}$ by

$$\gamma \mapsto h(\gamma).$$

By what has just been shown, $h$ is unbounded, in particular not identically 0. This has as a consequence that the homomorphism is non-trivial. This means that there must exist an undistorted, infinite order element $g$ (since any homomorphism into $\mathbb{Z}$ must annihilate finite order and distorted elements), with $h(g) \neq 0$. Without loss of generality we may assume $h(g) > 0$ (otherwise replacing $g$ by its inverse). But now by the metric spectral principle recalled above, there must exist a metric functional $h_1$ such that $h_1(g) \leq 0$. This shows that $h_1$ is different from $h$. Since $\tau(g) > 0$ the functional $h_1$ must take arbitrarily large negative values, showing that $h_1 \in \partial X$. □

In classical hyperbolic geometry, parabolic isometries are those which are distorted and preserve a horosphere. We say that an isometry $g$ of a metric space $X$ with base point $x_0$ is *monotone* if $d(x_0, g^n x_0) \to \infty$ monotonically for all sufficiently large $n$ as $n \to \infty$. (So one way of treating a general isometry $f$ might be to pass to a power of it, $g := f^N$.)

**Theorem 21.** *Let $g$ be a monotone isometry of a metric space $X$. If $\tau(g) = 0$, then there exists a metric functional that vanishes on the whole orbit $\{g^n x_0 : n \in \mathbb{Z}\}$.*

**Proof.** Let $Y$ be the orbit $\{g^n x_0 : n \in \mathbb{Z}\}$ with the metric induced by $X$. It is a proper metric space since $g$ is monotone. Consider the subset $A$ of $\partial Y$ of metric functionals $h$ for which $h(g^n x_0) \leq h(g^m x_0)$ for all $n \geq m$. The subset $A$ is closed since the inequalities pass to limits. It is also invariant under the group $H$ generated by $g$ since for any $n \geq m$,

$$\begin{aligned}(g.h)(g^n x_0) &= h(g^{n-1} x_0) - h(g^{-1} x_0) = h(g^{n-1} x_0) - h(g^{m-1} x_0) + (g.h)(g^m x_0) \\ &\leq (g.h)(g^m x_0).\end{aligned}$$

Next we verify that $A$ is non-empty. Take a converging subsequence of $h_{g^n x_0}$ as $n \to \infty$, in notation $h_{g^{n_i} x_0} \to h$. Then notice that by the monotonicity of $g$, for fixed $m \geq k$, we have

$$\begin{aligned} h(g^m x_0) &= \lim_i d(g^m x_0, g^{n_i} x_0) - d(x_0, g^{n_i} x_0) \\ &= \lim_i d(x, g^{n_i - m} x_0) - d(x_0, g^{n_i} x_0) \\ &\leq \lim_i d(x_0, g^{n_i - k} x_0) - d(x_0, g^{n_i} x_0) = h(g^k x_0). \end{aligned}$$

Since $H$ is a cyclic group acting on the compact non-empty set $A$ by homeomorphisms, there is an invariant probability measure $\mu$ on $A$. Therefore, as remarked above (with details found in [25, Proposition 2]),

$$T(g) = \int_{\partial Y} h(g x_0) d\mu(h)$$

defines a 1-Lipschitz group homomorphism $T : H \to \mathbb{R}$. Since any element of $\mathbb{R}$ is undistorted the image of $g$ must be 0, and so for every $n > 0$

$$0 = \int_{\partial Y} h(g^n x_0) d\mu(h).$$

On the other hand $h(g^n x_0) \leq h(x_0) = 0$. This implies that for every $n$ the set of $h$ for which $h(g^n x_0) = 0$ has full measure. The intersection of countable full measure sets has full measure, therefore there exists at least one $h$ which vanishes on the whole orbit, thus proving the theorem. □

**Remark 22.** Note that the function $h \equiv 0$ is a metric functional on any infinite dimensional Hilbert space, while as an additive group no element is distorted. This shows that no direct general converse holds.

Now we observe one thing in relation to the interesting notion of reduced boundary from [2] (note a conjecture in this paper that states that for finitely generated nilpotent groups all reduced boundary points should be fixed by the whole group). We extend their definition by also considering points in $X$ and in the weak topology. It is easy to verify that the action of isometries extends to the reduced metric boundary, with the following calculation: Say that $H$ is equivalent to $h$ differing by at most a constant $C$. Then

$$|gH(x) - gh(x)| = \left|H(g^{-1}x) - H(g^{-1}x_0) - h(g^{-1}x) + h(g^{-1}x_0)\right| \leq 2C.$$

**Proposition 23.** *Let $g$ be an isometry of a metric space. Then it fixes a point in the reduced metric compactification.*

**Proof.** Take a limit point $h$ of the orbit $g^n x_0$ which exists by compactness. Recall that $(g.h)(x) = h(g^{-1}x) - h(g^{-1}x_0)$. The last term is bounded so we can forget this when passing to the reduced compactification. We calculate:

$$\begin{aligned} d(g^{-1}x, g^n x_0) - d(x_0, g^n x_0) &\leq d(g^{-1}x, g^{n-1}x_0) + d(g^{n-1}x_0, g^n x_0) - d(x_0, g^n x_0) \\ &= d(g^{-1}x_0, x_0) + d(x, g^n x) - d(x_0, g^n x_0) \end{aligned}$$

for any $x \in X$. For any limit point $h$ this calculation gives $h(g^{-1}x) \leq d(g^{-1}x_0, x_0) + h(x)$. The reverse inequality is obtained by applying the inequality to $gx$ instead of $x$. Since the action by the isometry $g$ is a well-defined map of the reduced boundary, this proves the proposition. □

## 6. Concluding remarks

This short section provides some remarks and suggestions of preliminary nature.

**Weak topology and convex sets.** Recall that in Banach spaces weakly convergent sequences are bounded. Also if a weakly convergent sequence belong to a closed convex set, then the weak limit is also contained in this set (Mazur's theorem, see for example [30, p. 103]). Notice that in the example in Remark 13, $h_{x_n}$ converges weakly but is not bounded. On the other hand the limit belongs to the convex hull of the sequence. One could wonder if there is a generalization of this. In a yet different direction related to this, we refer the reader to [37].

**Dual space.** One could imagine defining the weak topology by declaring a sequence $x_n$ weakly converging to $x$ if $h(x_n) \to h(x)$ for every metric functional on $X$. But this would give back the usual strong, or metric, topology, since we could look at $h = h_x$. Then it would make more sense to only consider $h$ which are at infinity. (I am indebted to V. Guirardel and T. Hartnick for these remarks.) Now it could be interesting to see in what way these $h$s coordinate the space $X$ in the sense of looking at all the real values $h(x)$ for each fixed $x$. For example, if $X$ is a geodesic space with the property that every geodesic segment can be extended to a ray, then one would have

$$d(x_0, x) = \sup_h |h(x)|,$$

where the supremum is taken over all Busemann functions. Indeed, we take the geodesic from $x_0$ to $x$ and then extend it indefinitely. For the corresponding Busemann function $h$, one has $|h(x)| = d(x_0, x)$.

One could also try to define a norm on the Busemann functions (or metric functionals coming from unbounded sequences):

$$\|h\| = \sup_{x \neq x_0} \frac{|h(x)|}{d(x, x_0)}.$$

Note that this is not always 1, for example for infinite dimensional Hilbert spaces where all linear functionals of norm at most 1 are metric functionals, see [27]. With this one would have a metric on the "dual space" of a metric space. At times, I have also suggested the notion for a metric space to be reflexive: if every horofunction is a Busemann function.

**Differentiability in metric spaces.** Here is a remark from [27]: In the works of Cheeger and collaborators on differentiability of functions on metric spaces, the notion of generalized linear function appears. In [9] it is connected to Busemann functions, on the other hand that author remarks in [8] that non-constant such functions do not exist for most spaces. Perhaps it remains to investigate how metric functionals relate to this subject.

**Random walks.** It is shown above (and easily observed in case of abelian groups for example) that the metric boundary always contains at least two points. One may therefore conclude that it is not the Poisson boundary of random walks. But it is conceivable at least for some large classes of groups, that if we pass to the reduced boundary, then random walks could converge to points in this space (a recurrent, or drift zero, random walk could be said to converge to the interior which reduced is one point). In this way hitting probability measures on the reduced metric boundary could describe the behavior of random walks in a more refined way. For example, random walks on the integers are governed by the classical law of large numbers under a first moment condition. The expectation value can be negative, zero or positive, depending on whether the defining random walk measure is asymmetric or not. The reduced metric compactification (with respect to any generating set) consists of three points, that naturally can be denoted $-\infty$, $+\infty$ and all finite points, that is, $\mathbb{Z}$ itself. And these three points describe the asymptotic behavior of random walks with negative, positive, or no drift, respectively.

**Incidence geometry.** If the reduced boundary is not enough one could consider the star geometry, in the sense of [23], associated to the (reduced) metric compactification. This could be invariant of the chosen generating set for the Cayley graphs. It would also be of interest to study how groups act on this, their associated incidence geometry. This is related to the conjecture in [2] already mentioned above.

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