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Heat Kernels, Theta Identities, and Zeta Functions on Cyclic Groups

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Abstract. We prove a theta relation analogous to the classical Poisson–Jacobi theta inversion formula and deduce two formulas for the associated zeta functions. The proof is based on determinations of the heat kernel on $\mathbb{Z}$ and on $\mathbb{Z}/m\mathbb{Z}$. The theta identity gives in particular an interesting formula for certain sums of Bessel functions.

1. Introduction

The systematic way heat kernels give rise to zeta functions and their functional equations via theta inversion formulas has been emphasized by Jorgenson and Lang, see [**JL01a**] and [**JL01b**]. In the present paper we consider heat kernels on $\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}$ and derive a theta inversion formula involving a modified Bessel function of the first kind I_x. In fact, we will see that

$$e^{-t}I_x\left(t\right)$$

is the heat kernel on $\mathbb{Z}$ and thus plays a role similar to that of the Gaussian

$$\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}$$

on $\mathbb{R}$. Just as in the classical case one gets a deep formula when determining the heat kernel on a quotient in two ways. One way is through periodization of the heat kernel on the covering and the other way is through spectral theory on the quotient. The equality is guaranteed by the uniqueness of the heat kernel. Alternatively this can be proved by reducing it to an appropriate (generalized) Poisson summation formula. On $\mathbb{R}$ one gets the classical

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Poisson–Jacobi theta inversion formula:

$$\frac{1}{\sqrt{4\pi t}}\sum_{n=-\infty}^{\infty} e^{-\pi^2 n^2/t} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{-n^2 t}$$

for all $t > 0$. In our case we get:

THEOREM 1. *For $X = \mathbb{Z}$ and $\Gamma = m\mathbb{Z}$, the associated theta relation is*

$$e^{-t}\sum_{k=-\infty}^{\infty} I_{km}(t) = \frac{1}{m}\sum_{j=0}^{m-1} e^{-2\sin^2(\pi j/m)t},$$

for all t and where I_x is the modified Bessel function of the first kind.

More general such trace formulas were established by F. Chung and S.-T. Yau in [**CY97**]. If we instead focus on Bessel sums we get the following formula which does not seem to appear explicitly in the literature:

THEOREM 2. *For any $z \in \mathbb{C}$ and integers x and $m > 0$*

$$\sum_{k=-\infty}^{\infty} I_{x+km}(z) = \frac{1}{m}\sum_{j=0}^{m-1} e^{\cos(2\pi j/m)z+2\pi ijx/m}.$$

The sum on the left is an example of a Neumann series. The special case $x = 0$ was proved by Al-Jarrah, Dempsey, and Glasser in [**ADG02**]:

COROLLARY 3. *For any $z \in \mathbb{C}$ and integer $m > 0$*

$$\sum_{k=-\infty}^{\infty} I_{km}(z) = \frac{1}{m}\sum_{j=0}^{m-1} e^{\cos(2\pi j/m)z}.$$

The proof in [**ADG02**] is based on a Poisson summation formula due to Titchmarsh and is therefore perhaps less conceptual than our proof. The authors in [**ADG02**] deduce a number of corollaries from their beautiful formula.

The classical theta identity has many important applications in number theory, for example it leads to the functional equation and meromorphic continuation of Riemann's zeta function and to a proof of the Schaar–Landsberg formula which in turn implies Gauss quadratic reciprocity. In our case we are able to derive two formulas for associated zeta functions. The first one is:

THEOREM 4. *The Gauss transform of the theta relation on $\mathbb{Z}/m\mathbb{Z}$ gives for s, $\mathrm{Re}(s) \neq 0$, the following formula for the associated additive zeta* *function*

$$L^{\mathbb{Z}/m\mathbb{Z}}(s) := \frac{1}{m}\sum_{j=0}^{m-1} \frac{2s}{s^2 + 2\sin^2(\pi j/m)}$$
$$= \frac{2s}{\sqrt{s^4+2s^2}} \frac{1 + \left(s^2+1-\sqrt{s^4+2s^2}\right)^m}{1 - \left(s^2+1-\sqrt{s^4+2s^2}\right)^m}.$$

This can be compared with the classical case where the application of the Gauss transform yields

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \frac{2s}{s^2+n^2} = \frac{1+e^{-2\pi s}}{1-e^{-2\pi s}}$$

for s, $\mathrm{Re}(s) \neq 0$. Note that from this formula one computes $\zeta(2k)$ following Euler. Next we have:

THEOREM 5. *The corresponding Selberg-type zeta function turns out to be*

$$Z^{\mathbb{Z}/m\mathbb{Z}}(s) := \prod_{j=0}^{m-1} \left(s^2 + 1 - \cos(2\pi j/m)\right)$$
$$= 2^{2-m}\sinh^2\left(\frac{m}{2}\operatorname{Arcosh}\left(s^2+1\right)\right)$$

for any complex s. It has the obvious functional equation $Z^{\mathbb{Z}/m\mathbb{Z}}(s) = Z^{\mathbb{Z}/m\mathbb{Z}}(-s)$ and it has all its zeros on the imaginary axis.

Our Selberg-type zeta function can be defined for any finite graph and contains significant graph theoretical information. For example,

$$\left.\frac{Z(s)}{s^2}\right|_{s=0} = \prod_{j=1}^{m-1} \lambda_j,$$

which equals the number of spanning trees in the graph by a well-known fact due to Kirchoff. This important number measures the complexity of a graph, see [**Bi93**].

The present work gives an explicit example for the section "the heat kernel on totally disconnected or discrete spaces" in [**JL01a**]. In that same reference it is written that "the spectral decomposition of the heat kernel is a (the?) source of theta inversion formulas, and therefore of functional equations for the corresponding zeta function." Our paper thus exhibits a model case for further theta relations coming from other finitely generated groups.

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2. Preliminaries

The heat kernel on a graph. Let X be a graph and denote by $N(x)$ the set of vertices adjacent to x. The (combinatorial) Laplacian Δ on X is defined by

$$\Delta f(x) = f(x) - \frac{1}{|N(x)|} \sum_{y \in N(x)} f(y).$$

Thus the heat equation is

$$\left(\Delta + \frac{\partial}{\partial t}\right) f(t,x) = 0.$$

We define the delta function $\delta_y(x) = 0$ for $x \neq y$ and $\delta_y(y) = 1$. Fix a base vertex $0 \in X$. By the *heat kernel* $K^X(t,x)$ we mean the fundamental solution to the above equation, i. e. the solution of this equation with initial value $K^X(0,x) = \delta_0(x)$.

Let X be a finite graph with vertex set $\{0,1,2,\ldots,m-1\}$. The Laplacian is a symmetric matrix (in the basis $\{\delta_y\}$) with real eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{m-1}$ and corresponding orthonormal basis of eigenvectors $\varphi_0, \varphi_1, \ldots, \varphi_{m-1}$. We can express δ_0 in this basis:

$$\delta_0 = \sum_{j=0}^{m-1} \overline{\varphi_j(0)} \varphi_j.$$

The heat kernel is then equal to

$$K^X(t,x) = e^{-t\Delta}\delta_0(x) = \sum_{j=0}^{m-1} e^{-\lambda_j t} \varphi_j(x) \overline{\varphi_j(0)},$$

since $e^{-0\Delta}\delta_0 = \delta_0$ and for a function f on X (for example δ_0)

$$\left(\Delta + \frac{\partial}{\partial t}\right) e^{-t\Delta} f = \Delta e^{-t\Delta} f + \left(\frac{\partial}{\partial t} e^{-t\Delta}\right) f = \Delta e^{-t\Delta} f - \Delta e^{-t\Delta} f = 0.$$

For more information, see [**Ch96**].

The Cayley graph of a group. Let G be a group generated by a finite set S. Following Cayley one can then associate a graph X which has the elements of G as vertex set and where g and h are adjacent if there is an element s in $S \cup S^{-1}$ such that $g = hs$. For example, consider the group $\mathbb{Z}$ generated by 1, then (a geometric realization of) X is the real line with vertices at the integers. For a finite cyclic group $\mathbb{Z}/m\mathbb{Z}$, we take the same generator and obtain a graph which is a cycle of length m.

Modified Bessel functions. For integers $x \geq 0$ and complex numbers z, the modified Bessel function of the first kind is the function

$$I_x(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{x+2k}}{k!\,(x+k)!} = \frac{1}{\pi} \int_0^{\pi} e^{z\cos\vartheta} \cos x\vartheta \, d\vartheta$$

and for negative x, $I_x = I_{-x}$. It is a solution to the differential equation

$$z^2 \frac{d^2 w}{dz^2} + z\frac{dw}{dz} - \left(z^2 + x^2\right) w = 0$$

and it relates by $I_x(z) = e^{-\pi i x/2} J_x\left(z e^{\pi i/2}\right)$ to the ordinary J-Bessel function. One of the recurrence relations satisfied by I_x is ([**Wa66**, p. 79]):

$$I_{x+1} + I_{x-1} = 2I_x'$$

for all integers x. For large t one has the asymptotics ([**Je00**, 1.7.9])

$$I_\nu(t) \sim \frac{e^t}{\sqrt{2\pi t}}\left(1 - \frac{4\nu^2 - 1}{8t}\right).$$

3. The heat kernel on $\mathbb{Z}$

Our space is the infinite cyclic group $\mathbb{Z}$, or more precisely the Cayley graph of $\mathbb{Z}$ associated to the standard generator. The corresponding Laplacian is given by

$$\Delta f(x) = f(x) - \frac{1}{2}\left(f(x+1) + f(x-1)\right)$$

where $x \in \mathbb{Z}$. The heat equation

$$\left(\Delta + \frac{\partial}{\partial t}\right) f(t,x) = 0, \tag{1}$$

can be solved by first taking Fourier transform in the x variable, then solving the resulting ODE, and finally transforming the solution back. This gives the solution

$$(-1)^x \sum_{n=|x|}^{\infty} \frac{(-t/2)^n}{n!}\binom{2n}{n-x}$$

which can be seen (see Lemma 7 below) to be equal

$$K^{\mathbb{Z}}(t,x) := e^{-t} I_x(t),$$

where I_x is a modified Bessel function of the first kind.

One can actually verify directly that $e^{-t}I_x(t)$ is the heat kernel. First, it is clear from the formulas that $I_x(0) = 0$ unless $x = 0$ in which case it is 1. As for the heat equation we have

$$\begin{aligned}
&\left(\Delta + \frac{\partial}{\partial t}\right) e^{-t} I_x(t) \\
&= e^{-t} I_x(t) - \frac{1}{2}\left(e^{-t} I_{x+1}(t) + e^{-t} I_{x-1}(t)\right) - e^{-t} I_x(t) + e^{-t} I_x'(t) \\
&= -\frac{1}{2} e^{-t}\left(I_{x+1}(t) + I_{x-1}(t) - 2I_x'(t)\right) = 0,
\end{aligned}$$

where in the last step we used the recurrence relation recalled in the previous section. The fact that $e^{-t}I_x(t)$ is an analog of the Gaussian is further illustrated by the asymptotics

$$e^{-t}I_x(t) \sim \frac{1}{\sqrt{2\pi t}}\left(1 - \frac{4x^2-1}{8t}\right),$$

for large t.

REMARK 6. The probabilistic interpretation of equation (1) is a randomized symmetric simple random walk on $\mathbb{Z}$ where the jumps occur Poisson distributed with common density e^{-t}, see [**Fe71**, pp. 58–60] for more details although the heat equation is not explicitly mentioned there. For a recent paper on more general heat equations on $\mathbb{Z}$ we refer to [**GI02**].

4. The heat kernel on $\mathbb{Z}/m\mathbb{Z}$

The space is now the group $\mathbb{Z}/m\mathbb{Z}$ which can be viewed as a graph in the standard way i. e. the Cayley graph with generator set $\{1\}$ or, what amounts to the same for our purposes, $\{\pm 1\}$. The Laplacian is then given by

$$\Delta f(x) = f(x) - \frac{1}{2}\left(f(x+1) + f(x-1)\right)$$

where $x \in \mathbb{Z}/m\mathbb{Z}$.

Recall that for a finite abelian group with symmetric generating set S, character χ, and Laplacian

$$\Delta f(x) = f(x) - \frac{1}{|S|}\sum_{s\in S} f(x+s)$$

one has

$$\Delta\chi = \left(1 - \frac{1}{|S|}\sum_{s\in S}\chi(s)\right)\chi.$$

In other words, χ is an eigenfunction. In our case $S = \{\pm 1\}$ and the characters are

$$\chi_j(x) = e^{2\pi i j x/m}$$

with $1 - \cos(2\pi j/m)$ as corresponding eigenvalues. This implies

$$e^{-t\Delta}\chi_j = e^{-t(1-\cos(2\pi j/m))}\chi_j.$$

We will use the identity $1 - \cos 2\alpha = 2\sin^2\alpha$. By orthogonality relations, δ_y can be written

$$\delta_y = \frac{1}{m}\sum_{j=0}^{m-1}\overline{\chi_j(y)}\chi_j.$$

The heat kernel is thus

$$K^{\mathbb{Z}/m\mathbb{Z}}(t,x) := e^{-t\Delta}\delta_0(x) = \frac{1}{m}\sum_{j=0}^{m-1} e^{-2\sin^2(\pi j/m)t}e^{2\pi i j x/m},$$

as explained in the preliminaries.

5. A theta inversion formula

The periodization of $K^{\mathbb{Z}}$ by the subgroup $m\mathbb{Z}$ yields a periodic function on $\mathbb{Z}$ or alternatively a function on the quotient $\mathbb{Z}/m\mathbb{Z}$:

$$e^{-t}\sum_{k=-\infty}^{\infty} I_{x+km}(t).$$

This converges for every t because

$$\begin{aligned}\left|\sum_{k=0}^{\infty} I_{x+km}(t)\right| &\leq \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{|t/2|^{x+km+2n}}{n!\,(x+km+n)!}\\ &\leq \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{|t/2|^{k+n}}{n!k!}\leq \exp(|t|)\end{aligned}$$

for $x \geq 0$. Since the Laplacian has the same form this will solve the heat equation on $\mathbb{Z}/m\mathbb{Z}$ and note also that this is equal to $\delta_0(x)$ for $t = 0$. So we now claim that

$$e^{-t}\sum_{k=-\infty}^{\infty} I_{x+km}(t) = \frac{1}{m}\sum_{j=0}^{m-1} e^{-2\sin^2(\pi j/m)t}e^{2\pi ijx/m}. \tag{2}$$

This follows from the uniqueness of the heat kernel, but we will prove it directly by the following two lemmas.

LEMMA 7. *The xth Fourier coefficient of $\omega \mapsto \exp\left(-2\sin^2(\omega/2)\,t\right)$ has the Taylor series expansion*

$$(-1)^x\sum_{n=|x|}^{\infty}\frac{(-t/2)^n}{n!}\binom{2n}{n-x}$$

which in fact is equal to $e^{-t}I_x(t)$.

PROOF. First note that

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}\exp\left(-2\sin^2(\omega/2)\,t\right)e^{-i\omega x}\,d\omega\\ &=\frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{(-2t)^n}{n!}\int_0^{2\pi}\sin^{2n}(\omega/2)\,e^{-i\omega x}\,d\omega\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\sin^{2n}(\omega/2)\,e^{-i\omega x}\,d\omega\\
&=\frac{1}{2\pi}\sum_{y=0}^{2n}\binom{2n}{y}(-1)^y(2i)^{-2n}\int_0^{2\pi}e^{i\omega(n-y)}e^{-i\omega x}\,d\omega\\
&=\begin{cases}(-1)^x4^{-n}\binom{2n}{n-x} & \text{if } n\geq|x|\\ 0 & \text{if } n\leq|x|-1\end{cases}.
\end{aligned}$$

Thus we obtain the Fourier coefficient

$$(-1)^x\sum_{n=|x|}^{\infty}\frac{(-t/2)^n}{n!}\binom{2n}{n-x}.$$

Furthermore $\exp\left(-2\sin^2(\omega/2)\,t\right)=e^{-t}\exp(t\cos\omega)$ and

$$\int_0^{2\pi}\exp(t\cos\omega)\,e^{-i\omega x}\,d\omega=2\int_0^{\pi}\exp(t\cos\omega)\cos(\omega x)\,d\omega=2\pi I_x(t).$$

This proves the lemma. □

Next we need:

Lemma 8. *The jth Fourier coefficient of*

$$x\mapsto\sum_{k=-\infty}^{\infty}(-1)^{x+km}\sum_{n=|x+km|}^{\infty}\frac{(-t/2)^n}{n!}\binom{2n}{n-x-km}$$

is $\exp\left(-2t\sin^2\pi j/m\right)$.

Proof. We have

$$\begin{aligned}
&\sum_{x=0}^{m-1}\sum_{k=-\infty}^{\infty}(-1)^{x+km}\sum_{n\geq|x+km|}\frac{(-t/2)^n}{n!}\binom{2n}{n-x-km}e^{-2\pi ixj/m}\\
&=\sum_{y=-\infty}^{\infty}(-1)^y\sum_{n\geq|y|}\frac{(-t/2)^n}{n!}\binom{2n}{n-y}e^{-2\pi iyj/m}\\
&=\sum_{n=0}^{\infty}\frac{(-t/2)^n}{n!}\sum_{y=-n}^{n}\left(-e^{-2\pi ij/m}\right)^y\binom{2n}{n-y}\\
&=\sum_{n=0}^{\infty}\frac{(-t/2)^n}{n!}(-1)^n\left(e^{\pi ij/m}-e^{-\pi ij/m}\right)^{2n}\\
&=\sum_{n=0}^{\infty}\frac{(-2t)^n}{n!}\sin^{2n}\pi j/m=\exp\left(-2t\sin^2\pi j/m\right).
\end{aligned}$$

□

Specializing to $x=0$ in (2) we so obtain our theta inversion formula:

THEOREM 9. *For every integer $m > 0$ and all t we have that*

$$e^{-t}\sum_{k=-\infty}^{\infty} I_{km}(t) = \frac{1}{m}\sum_{j=0}^{m-1} e^{-2\sin^2(\pi j/m)t}.$$

This form of the formula is analogous to the classical inversion formula. One can also cancel the factor e^{-t} in (2) and since both sides clearly are analytic functions converging for t in the whole complex plane we can formulate the following version which focuses more on the Bessel sum:

THEOREM 10. *For any $z \in \mathbb{C}$ and integers x and $m > 0$*

$$\sum_{k=-\infty}^{\infty} I_{x+km}(z) = \frac{1}{m}\sum_{j=0}^{m-1} e^{\cos(2\pi j/m)z+2\pi ijx/m}.$$

We could not find these beautiful formulas in the classical literature, including books and tables on Bessel functions. For example Doetsch [**Do34**] studied Bessel sums but in the t-variable. The only formulas we have found in tables of series concern $m = 1$ and $m = 2$ ([**Ha75**, p. 411]):

$$\sum_{k=1}^{\infty} I_k(z) = \frac{1}{2}e^z - \frac{1}{2}I_0(z) \text{ and}$$
$$\sum_{k=0}^{\infty} I_{2k}(z) = \frac{1}{2}\cosh z + \frac{1}{2}I_0(z),$$

which immediately can be verified to be the same as the corresponding special cases of Theorem 10 or Corollary 3 just as in [**ADG02**].

In any case it is not the main point of this article to prove new formulas for Bessel series but rather to exhibit an example belonging to an important structural context, thus emphasizing the relevance of this formula for applications and to gain intuition for more complicated situations.

REMARK 11. Note the usual feature of formulas such as the one in Theorem 9: when t is near 0 then the left sum is essentially only one term, while not so on the right, and vice versa for large t. In our case an additional striking feature is of course that one sum has an infinite number of terms while the other one is a finite sum. As in the classical case this is an explicit case of a generalized Poisson summation formula, see [**Fo95**, Theorem 4.42]. Note also that the sum on the right is a finite analog of a Bessel, just as Kloosterman sums are, but in another way.

6. A simple special case of the theta formula

The purpose of this section is to illustrate our theta identity by deriving a consequence of it. Since the formula in Corollary 3 holds for all t and the sum is absolutely convergent, one could differentiate in t termwise and put

$t = 0$. We start with

$$I_0(t) + 2\sum_{k=1}^{\infty} I_{km}(t) = \frac{1}{m}\sum_{j=0}^{m-1} e^{\cos(2\pi j/m)t},$$

differentiate it n times and set $t = 0$:

$$I_0^{(n)}(0) + 2\sum_{k=1}^{\infty} I_{km}^{(n)}(0) = \frac{1}{m}\sum_{j=0}^{m-1} \cos^n(2\pi j/m).$$

Recall that for $km \geq 0$,

$$I_{km}(t) = \sum_{l=0}^{\infty} \frac{t^{km+2l}}{2^{km+2l} l!\,(km+l)!}.$$

Differentiating termwise and letting $t = 0$ we obtain

$$I_{km}^{(n)}(0) = \frac{n!}{2^n l!\,(n-l)!}$$

provided $km + 2l = n$ otherwise it is 0. Hence we have:

COROLLARY 12. *The following formula holds:*

$$\sum_{j=0}^{m-1} \cos^n(2\pi j/m) = \frac{m}{2^n}\sum_{l\in L(n,m)} \binom{n}{l},$$

where $L(n,m)$ denotes the set of integers l, with $0 \leq l \leq n$ of the form $(n - mq)/2$ for some integer q. In particular, if n is odd and m even, then $L(n,m) = \emptyset$ and the sum is 0.

For example, if $n = 2$ and $m \geq 3$ one has

$$\sum_{j=0}^{m-1} \cos^2(2\pi j/m) = \frac{m}{2}.$$

These formulas can be found in tables of formulas. The standard way of proving them is to write $\cos\alpha = \left(e^{i\alpha} + e^{-i\alpha}\right)/2$ and expand the terms when taking powers using the binomial theorem and then simplify. We could also get a similar formula for sine starting with the other version of the theta formula.

7. Zeta functions

Riemann took the Mellin transform of the classical theta function and used the Poisson–Jacobi theta inversion formula to obtain the functional equation and meromorphic continuation of the Riemann zeta function. Quite generally, the Mellin transform of a theta function gives you a spectral zeta. Alternatively, one can take the Gauss transform as Jorgenson–Lang suggest and obtain an additive zeta, which is the logarithmic derivative of a Selberg-type zeta, cf. [**Mc72**] and [**JL01b**].

The Mellin transform of a function f is

$$(\mathbf{M}f)(s) = \int_0^\infty f(t)\, t^s \, \frac{dt}{t}$$

whenever it converges. For example, Mellin of e^{-t} is the gamma function Γ. If we Mellin transform the right hand side in Theorem 9 we get, after a regularization which amounts to subtracting the constant term,

$$\Gamma(s)\,\zeta(2s),$$

where

$$\zeta(s) := \frac{1}{m}\sum_{j=1}^{m-1} \frac{1}{\left(\sqrt{2}\sin(\pi j/m)\right)^s}.$$

Although there is a formula for Mellin of the terms on the left hand side, namely

$$\mathbf{M}\left(e^{-t} I_x(t)\right)(s) = \frac{\Gamma(s+x)\,\Gamma(1/2-s)}{2^s \pi^{1/2}\Gamma(1+x-s)}$$

for $-\operatorname{Re}(x) < \operatorname{Re}(s) < 1/2$, we are not able to get a reasonable transformed left hand side.

Instead we take the Gauss transform

$$(\mathbf{G}f)(s) = 2s\int_0^\infty f(t)\, e^{-s^2 t}\, dt,$$

which essentially is a Laplace transform. Applied to the right side in Theorem 9 we get (no need to regularize)

$$L(s) := \frac{1}{m}\sum_{j=0}^{m-1} \frac{2s}{s^2 + 2\sin^2(\pi j/m)}.$$

Now let (we continue to suppress the dependence on m in the notation)

$$Z(s) = \prod_{j=0}^{m-1}\left(s^2 + 1 - \cos(2\pi j/m)\right),$$

which has a functional equation from the obvious symmetry $s \leftrightarrow -s$. Then

$$\frac{Z'(s)}{Z(s)} = mL(s).$$

At this point it is worth remarking that Z is rather reminiscent of the Ihara zeta function (see [**ST96**] and [**Te99**]) of the graph $\mathbb{Z}/m\mathbb{Z}$:

$$Z^{\mathrm{Ih}}(s) = (1-s^m)^{-2} = \prod_{j=0}^{m-1}\left(s^2 + 1 - 2s\cos(2\pi j/m)\right)^{-1}.$$

Here the first equality comes from the definition in terms of lengths of equivalence classes of closed paths, while the second equality comes from Ihara's formula.

In any case, the Gauss transform of individual terms on the left hand side in Theorem 9 are for $\mathrm{Re}(s) \neq 0$ and $n \geq 0$

$$\mathbf{G}\left(e^{-t} I_n(t)\right)(s) = \frac{2s\left(s^2+1-\sqrt{s^4+2s^2}\right)^n}{\sqrt{s^4+2s^2}}.$$

The sum

$$\mathbf{G}\left(e^{-t} \sum_{k=-\infty}^{\infty} I_{km}(t)\right)(s) = \sum_{k=-\infty}^{\infty} \mathbf{G}\left(e^{-t} I_{km}(t)\right)(s)$$

is therefore geometric and sums to the following expression after some algebraic simplifications

$$\frac{2s}{\sqrt{s^4+2s^2}} \frac{1+\left(s^2+1-\sqrt{s^4+2s^2}\right)^m}{1-\left(s^2+1-\sqrt{s^4+2s^2}\right)^m}.$$

Hence we have proved Theorem 4 of the introduction.

Now notice that the logarithmic derivative of

$$C\left(\left(s^2+1+\sqrt{s^4+2s^2}\right)^m + \left(s^2+1-\sqrt{s^4+2s^2}\right)^m - 2\right)$$

equals $mL(s)$. After pinning down the undetermined constant C to be 2^{-m} as

$$\left(s^2+1+\sqrt{s^4+2s^2}\right)^m + \left(s^2+1-\sqrt{s^4+2s^2}\right)^m$$
$$= 2\sum_{l=0}^{[m/2]} \binom{m}{2l} \left(s^2+1\right)^{m-2l} \left(s^4+2s^2\right)^l = 2^m s^{2m} + P\left(s^2\right)$$

where $[m/2]$ is the largest integer less than or equal to $m/2$ and P is some polynomial of degree strictly less than m one has for all s that

$$Z(s) = 2^{-m}\left(\left(s^2+1+\sqrt{s^4+2s^2}\right)^m + \left(s^2+1-\sqrt{s^4+2s^2}\right)^m - 2\right)$$

which can be verified to equal

$$2^{2-m} \sinh^2\left(\frac{m}{2} \operatorname{Arcosh}\left(s^2+1\right)\right).$$

This proves the formula in Theorem 5.

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