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Two extensions of Thurston's spectral theorem for surface diffeomorphisms

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Abstract

Thurston obtained a classification of individual surface homeomorphisms via the dynamics of the corresponding mapping class elements on Teichmüller space. In this paper, we present certain extended versions of this, first, to random products of homeomorphisms and, second, to holomorphic self-maps of Teichmüller spaces.

1. *Introduction*

Let M be an oriented closed surface of genus $g \geqslant 2$. Let $\mathcal{S}$ denote the isotopy classes of simple closed curves on M not isotopically trivial. For a Riemannian metric ρ on M and a closed curve β, let $l_\rho(\beta)$ be the infimum of the length of curves isotopic to β.

In a seminal preprint from 1976 [**31**], Thurston classified surface diffeomorphisms as being isotopic either to a periodic one, or else reducible or pseudo-Anosov. A version of this was obtained earlier in a series of papers by Nielsen [**25**, **26**]. Using the theory of foliations of surfaces, Thurston showed the following consequence; the proof is worked out in exposé 11 of [**9**, Théorème Spectral]:

Theorem 1 ([**31**, Theorem 5]). *For any diffeomorphism f of M, there is a finite set $1 \leqslant \lambda_1 < \lambda_2 < \ldots < \lambda_K$ of algebraic integers such that, for any $\alpha \in \mathcal{S}$, there is a λ_i such that, for any Riemannian metric ρ,*

$$\lim_{n\to\infty} l_\rho(f^n\alpha)^{1/n} = \lambda_i.$$

The map f is isotopic to a pseudo-Anosov map if and only if $K = 1$ and $\lambda_1 > 1$.

This statement is analogous to the dynamical behaviour of linear maps A of finite-dimensional vector spaces: the limits $\lim_{n\to\infty} \|A^n v\|^{1/n}$ exist for every vector v, as is immediate from the Jordan normal form. In this note, we will obtain a few partial extensions of Theorem 1. First, we have the following theorem.

Theorem 2. *For any integrable ergodic cocycle $f_n = g_n g_{n-1} \ldots g_1$ of orientation-preserving homeomorphisms of M, there are almost surely a constant $\lambda \geqslant 1$ and a (random) measured foliation μ such that, for any $\alpha \in \mathcal{S}$ with $i(\mu, \alpha) > 0$ and Riemannian metric ρ,*

$$\lim_{n\to\infty} l_\rho(f_n\alpha)^{1/n} = \lambda.$$

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Let $\mathcal{T}$ be the Teichmüller space of M and, for $x \in \mathcal{T}$, the corresponding hyperbolic length of $\alpha \in \mathcal{S}$ is denoted by $l_x(\alpha)$. The previous theorem is a direct consequence of the more precise statement.

THEOREM 3. *In the setting of the previous theorem, we have almost surely that, for any $x \in \mathcal{T}$, there is an explicit constant $C(\mu, x) > 0$ such that, for any $\epsilon > 0$, there is a number N for which*

$$C(\mu, x) i(\mu, \alpha)(\lambda - \epsilon)^n \leqslant l_x(f_n \alpha) \leqslant l_x(\alpha)(\lambda + \epsilon)^n$$

holds for any $\alpha \in \mathcal{S}$ and any $n > N$.

Note in particular the uniformity in α. Another aspect is the asymmetry of the inequalities: on the left-hand side the intersection number $i(\mu, \alpha)$ appears, which means that we have, as expected, no lower bound in the case $i(\mu, \alpha) = 0$. The constant $C(\mu, x)$ takes into account the discrepancy between the length of α at x and the intersection $i(\mu, \alpha)$, so that in particular the left-hand side is independent of multiplying μ with a positive constant as it has to be.

The mapping class group $\mathrm{MCG}(M)$ is the group of isotopy classes of orientation-preserving homeomorphisms (or diffeomorphisms) of M:

$$\mathrm{MCG}(M) = \mathrm{Homeo}^+(M)/\mathrm{Homeo}_0(M),$$

which acts by automorphisms of $\mathcal{T}(M)$. Thus, every random product of homeomorphism gives rise to a random product of mapping classes and acts on $\mathcal{T}$. Kaimanovich and Masur [**13**] studied random walks on MCG. They proved that if the support of the random walk measure generates a non-elementary subgroup, then almost every (a.e.) trajectory converges to points in the set of uniquely ergodic foliations of $\mathcal{PMF}$, which is the space of projective measured foliations. Taking this into account, we can deduce the following corollary.

COROLLARY 4. *Let $f_n = g_n g_{n-1} \dots g_1$ be a product of random homeomorphisms where g_i are chosen independently and distributed with a probability measure of finite first moment and that generates a subgroup containing two independent pseudo-Anosov maps. Then there is a number $\lambda > 1$ such that almost surely* (a.s.), *for any $\alpha \in \mathcal{S}$ and metric ρ,*

$$\lim_{n \to \infty} l_\rho(f_n \alpha)^{1/n} = \lambda.$$

This can be viewed as analogous to a well-known theorem of Furstenberg and to Oseledets' multiplicative ergodic theorem for random products of matrices. Information about these results and other references to the vast literature on random dynamical systems can be found in [**2**].

Given a complex structure x on M, the extremal length of an isotopy class of curve α is

$$\mathrm{Ext}_x(\alpha) = \sup_{\rho \in [x]} \frac{l_\rho(\alpha)^2}{\mathrm{Area}(\rho)},$$

where the supremum is taken over all metrics in the conformal class of x. Miyachi [**23**] noted that a normalized version of the extremal length function, denoted by E_P, extends continuously to the whole Gardiner–Masur compactification; see Section 3 below for more details. A boundary point P is called *uniquely ergodic* if $E_P(\beta) > 0$ for all $\beta \in \mathcal{S}$.

By a theorem of Royden from 1971, later extended by Earle–Kra to surfaces M with punctures, the mapping class group, with some lower genus exceptions, is isomorphic to the complex automorphism group of $\mathcal{T}(M)$. Here is a statement about more general holomorphic self-maps, thus in a sense providing a certain extension of Theorem 1:

THEOREM 5. *Let $f : \mathcal{T} \to \mathcal{T}$ be a holomorphic map and $x \in \mathcal{T}$. Then there are a number $\lambda \geqslant 1$ and a limit point P of the orbit $\{f^n(x)\}_{n>0}$ in the Gardiner–Masur compactification such that, for all $n \geqslant 1$ and any curve $\beta \in \mathcal{S}$,*

$$\mathrm{Ext}_{f^n x}(\beta) \geqslant \left(\inf_\alpha \frac{\mathrm{Ext}_x^{1/2}(\alpha)}{E_P(\alpha)} \right)^2 E_P(\beta)^2 \lambda^n$$

and, provided that $E_P(\beta) > 0$,

$$\mathrm{Ext}_{f^n x}(\beta)^{1/n} \to \lambda.$$

The following can be seen as a weak generalization of the Nielsen–Thurston classification of mapping classes to general holomorphic self-maps of Teichmüller spaces.

THEOREM 6. *Let $f : \mathcal{T} \to \mathcal{T}$ be a holomorphic map. Then either every orbit in $\mathcal{T}$ is bounded, or every orbit leaves every compact set and there are associated points P (certain limit points of the orbit) in the Gardiner–Masur boundary. If P is uniquely ergodic, then it is unique and every orbit converges to this point in either compactification and for some $\lambda \geqslant 1$, any $\alpha \in \mathcal{S}$ and any $x \in \mathcal{T}$*

$$\mathrm{Ext}_{f^n x}(\alpha)^{1/n} \to \lambda \quad \text{and} \quad \inf_\alpha \frac{\mathrm{Ext}_{f(x)}^{1/2}(\alpha)}{E_P(\alpha)} \geqslant \lambda \inf_\alpha \frac{\mathrm{Ext}_x^{1/2}(\alpha)}{E_P(\alpha)}.$$

This classifies f as either having bounded orbits or else having certain associated boundary points P ('reducible' vs pseudo-Anosov depending on the nature of P). This is reminiscent of the Wolff–Denjoy theorem in complex dynamics that, together with a theorem of Fatou, classifies holomorphic self-maps of the unit disc. The P can informally be regarded as 'virtual fixed points at infinity'. Examples of important holomorphic self-maps beyond the automorphisms are the Thurston skinning map in three-dimensional topology and the Thurston pull-back maps in complex dynamics; see [**22**, **28**, **29**] for more details.

Method of proof: For the ergodic statements, the techniques of Ledrappier and myself [**17**] are being employed, although extended to asymmetric metrics, such as Thurston's Lipschitz metric. Together with the new insights about horofunctions by Walsh [**33**], this leads to the main result, Theorem 3. The proof of the corollary relies in addition on [**13**].

For Theorems 5 and 6, the starting point is the well-known fact that the Techmüller metric coincides with the Kobayashi metric, which implies that holomorphic maps are 1-Lipschitz in this metric (this is the only way in which the holomorphy is being used), after that the proof uses recent results on horofunctions [**14**, **19**, **23**, **24**].

Further comments:

(i) While Nielsen studied lifts of the diffeomorphisms to the hyperbolic disc and boundary circle, Thurston instead compactified $\mathcal{T}$ in a natural way, by adding $\mathcal{PMF}$, and applied Brouwer's fixed point theorem. Bers gave an alternative approach using more classical theory, in particular the Teichmüller metric [**3**]. For the spectral theorem (Theorem 1) Thurston used foliation theory; see [**9**] and also [**7**, § 14.5]. Our approach here instead uses two metrics on $\mathcal{T}$: Thurston's and Teichmüller's. In this context, it might be useful to point out that the recent paper [**18**] contains a study parallel to Bers' paper but for the Thurston metric.

(ii) Following the work of Kaimanovich–Masur [**13**, **21**] it is natural to wonder about so-called ray approximation; see [**12**], where this problem is mentioned. The first result in this direction is a paper of Duchin [**5**] which established ray approximation in the Teichmüller metric for times when the ray visits the thick part. Recently, Tiozzo [**33**] found an elegant argument removing the this part requirement. This could perhaps provide another approach to Corollary 4. Ray approximation for Weil-Petersson geodesics in the more general ergodic setting follows from the arguments in a paper by Margulis and myself; see [**16**]. A recent analysis of the harmonic measure can be found in [**10**].

(iii) During the last few years there has been significant progress in proving that almost every element in MCG is pseudo-Anosov, owing to the works of Maher, Rivin, Kowalski, and Lubotzky-Meiri; see, for example, [**20**]. It is perhaps interesting to compare this with Corollary 4, which states that random walk trajectories eventually look pseudo-Anosov from the perspective of Theorem 1.

(iv) Looking at Theorem 1 or Oseledets' theorem in the linear case, one might hope for a more precise statement than Theorem 2. There are, however, some obvious obstructions to keep in mind. The exponent λ can, in general, take any real value $\geqslant 1$ since one can take an appropriately asymmetric measure on a single pseudo-Anosov and its inverse. Moreover, for predicting which α grows exponentially in length, one should bear in mind the case of a coboundary $g(x) = f(\omega)f(T\omega)^{-1}$ and that the cocycle might consist of reducible maps with respect to one and the same curve system.

(v) Thurston used iteration on $\mathcal{T}$ in order to prove invariant structures [**22**], for example the existence of a hyperbolic three-manifolds fibering over the circle. Using [**13**], Farb and Masur showed that every homomorphism from higher rank lattices into MCG has finite image; see [**8**] for details. One could hope that the present results also will find application.

(vi) Our approach might be relevant for the study of random walks on $Out(F_n)$; the problem of extending the Kaimanovich–Masur theory is a well-known research problem.

(vii) As already indicated, one could moreover hope that our weak Wolff–Denjoy analogue could shed some light on the dynamics of rational maps in complex dynamics; the so-called Thurston obstruction theorem also leads to considering the iteration of a holomorphic self-map such as Thurston's pull-back maps. In this latter context the fixed point corresponds to the conformal map being combinatorially equivalent to a rational map. This is not always the case, and the conditions for checking this, that is, verifying that the holomorphic self-map has bounded orbit, are difficult in practice. The curves α for which $E_P(\alpha) = 0$ might be tightly related to the Thurston obstruction.

(viii) Does every holomorphic self-map of Teichmüller space with bounded orbit have a fixed point? A positive answer to this question would be a certain extension of Nielsen realization and also strengthen the analogy between Theorem 6 and the Wolff–Denjoy theorem.[†]

2. *Proof of Theorem* 3

Let (Ω, p) be a standard Borel space with $p(\Omega) = 1$ and $T : \Omega \to \Omega$ an ergodic measure-preserving transformation. Assume that $g : \Omega \to \text{Homeo}^+(M) \twoheadrightarrow \text{MCG}(M)$ is a measurable map and let

$$Z_n(\omega) = g(\omega)g(T\omega)\dots g(T^{n-1}\omega),$$

which is called an *ergodic cocycle*. Note here that we have shifted the order, so that in the terminology of the theorem, $f_n = Z_n^{-1}$ and the $g_i = (g(T^{i-1}\omega))^{-1}$. A *random walk* on MCG is the special case when the increments $g(T^i\omega)$ are assumed to be independent and identically distributed (this is precisely the case when (Ω, p) is a product space of infinite copies of a fixed probability space and T the shift).

[†] I mentioned this question in a seminar talk at Harvard, and a few months later it was answered affirmatively in [**1**] by Antonakoudis who attended the seminar.

For $x \in \mathcal{T}$ denote by $l_x(\alpha)$ the minimal length in its isotopy class in the hyperbolic metric x (or more precisely the class of isometric metrics all mutually isotopic). Let us recall Thurston's asymmetric Lipschitz metric [**30**],

$$L(x, y) = \log \sup_{\alpha \in \mathcal{S}} \frac{l_y(\alpha)}{l_x(\alpha)}.$$

It is easy to see that L verifies the triangle inequality, and it also true that it separates points although this is non-trivial. Therefore, L satisfies all the axioms for a metric except the symmetry, which indeed fails except in very special cases of surfaces with symmetries. The triangle inequality reads

$$L(x, z) \leqslant L(x, y) + L(y, z).$$

The metric is clearly invariant: $L(gx, gy) = L(x, y)$ for $g \in \mathrm{MCG}$. The topology induced by L coincides with the usual one; see [**27**].

Fix a base point $x_0 \in \mathcal{T}$. We will assume the following finite first moment condition for asymmetric metrics,

$$\int_\Omega (L(g(\omega)x_0, x_0) + L(x_0, g(\omega)x_0))\, dp(\omega) < \infty,$$

in which case we refer to f_n as an *integrable ergodic cocycle*, or informally as a *random product* of mapping class elements.

One has the following subadditivity property:

$$\begin{aligned} L(f_{n+m}(\omega)x_0, x_0) &\leqslant L(f_n(\omega)f_m(T^n\omega)x_0, f_n(\omega)x_0) + L(f_n(\omega)x_0, x_0) \\ &= L(f_m(T^n\omega)x_0, x_0) + L(f_n(\omega)x_0, x_0). \end{aligned}$$

From the subadditive ergodic theorem of Kingman one then knows that, for a.e. ω, the following limit exists (the finite non-negative value being independent of ω by the ergodicity assumption):

$$l := \lim_{n\to\infty} \frac{1}{n} L(f_n(\omega)x_0, x_0) = \lim_{n\to\infty} \frac{1}{n} L(x_0, Z_n(\omega)x_0).$$

We now introduce concepts from the work of Walsh [**33**], which gave a crucial inspiration for the present paper. It will be important to consider functions h in the so-called horofunction compactification of $\mathcal{T}$; that is, for $\mu \in \mathcal{PMF}$,

$$h_\mu(x) = \log \sup_\alpha \frac{i(\mu, \alpha)}{l_x(\alpha)} - \log \sup_\beta \frac{i(\mu, \beta)}{l_{x_0}(\beta)},$$

(note that it is well-defined for projective equivalence classes of measured foliations μ) and, for $x_n \to \mu$ in the Thurston compactification, one has

$$h_\mu(x) = \lim_{n\to\infty} L(x, x_n) - L(x_0, x_n).$$

These functions together with $h_z(x) = L(x, z) - L(x_0, z)$ constitute a compact space H homeomorphic to Thurston's compactification as proved by Walsh. Note that

$$h_z(x) = L(x, z) - L(x_0, z) \leqslant L(x, x_0)$$

by the triangle inequality.

It is a general fact that the action of the group of isometries extends to an action by homeomorphism of the horofunction boundary. The action is given by

$$(g.h)(x) = h(g^{-1}x) - h(g^{-1}x_0)$$

for an isometry g and a horofunction h.

For $f \in \mathrm{MCG}$ and h as above let $F(g,h) = -h(g^{-1}x_0)$. We note the following cocycle property:

$$F(g_1, g_2h) + F(g_2, h) = -(g_2 \cdot h)(g_1^{-1}x_0) - h(g_2^{-1}x_0)$$
$$= -h(g_2^{-1}g_1^{-1}x_0) + h(g_2^{-1}x_0) - h(g_2^{-1}x_0) = F(g_1g_2, h).$$

Note that, moreover,

$$L(gx_0, x_0) = -L(g^{-1}x_0, g^{-1}x_0) + L(x_0, g^{-1}x_0) = \max_{h \in H} F(g,h),$$

as follows from the triangle inequality. Following [**16**] (see also [**17**]), one introduces the so-called skew product system $\bar{T} : \Omega \times H \to \Omega \times H$ defined by $\bar{T}(\omega, h) = (T\omega, g^{-1}(\omega)h)$ and checks that, with $\bar{F}(\omega, h) := F(g(\omega)^{-1}, h)$, one has from the cocycle property above that

$$F(f_n(\omega)^{-1}, h) = \sum_{i=0}^{n-1} \bar{F}(\bar{T}^i(\omega, h)). \tag{1}$$

Moreover, we have $|F(g^{-1}(\omega), h)| \leqslant \max\{L(x_0, g(\omega)x_0), L(g(\omega)x_0, x_0)\}$ so F is integrable (note here again a small difference due to the asymmetric nature of L). From this point on, the proof runs as in the references mentioned, that is, a special measure is constructed which accounts for the drift as well as projecting to the original one. In some more detail, we would like to have a probability measure on $\Omega \times H$ which is preserved under $\bar{T}$ and projects onto the original measure p on the first factor, such that it accounts for the drift l in the following sense:

$$\int_{\Omega \times H} \bar{F} dm = l. \tag{2}$$

The measure m can be taken to be a weak limit point of

$$\frac{1}{n} \sum_{i=0}^{n-1} (\bar{T}^i)_* \mu_n$$

in the space of measures, and where μ_n is defined via

$$\int_{\Omega \times H} G(\omega, h)\, d\mu_n(\omega, h) = \int_{\Omega} G(\omega, h_{Z_n(\omega)x_0})\, dp(\omega)$$

for any integrable function G. By a standard argument one can moreover assume that this measure is ergodic. Birkhoff's ergodic theorem is applied and a measurable section is taken. In view of the definition of F and equalities (1), (2), we then get that, for a.e. ω, there is an $h = h^\omega$ such that

$$\lim_{n \to \infty} -\frac{1}{n} h(Z_n x_0) = l. \tag{3}$$

If $l > 0$, which is the non-trivial case, it is immediate that h must be a boundary point; indeed h_x for $x \in \mathcal{T}$ is bounded from below as is evident for the definition. Hence, by Walsh's theorem cited above, it is of the form h_μ for some $\mu \in \mathcal{PMF}$. This means that, for every $\epsilon > 0$, there is an N such that, for all $n > N$, one has

$$\log \sup_\alpha \frac{i(\mu, \alpha)}{l_{Z_n x_0}(\alpha)} - \log \sup_\beta \frac{i(\mu, \beta)}{l_{x_0}(\beta)} \leqslant -(l - \epsilon)n.$$

Letting $C_\mu^{-1} = \sup(i(\mu, \beta)/l_{x_0}(\beta))$ we then obtain

$$\sup_\alpha \frac{i(\mu, \alpha)}{l_{Z_n x_0}(\alpha)} \leqslant C_\mu^{-1}\, e^{-(l-\epsilon)n},$$

which leads to that, for every α, we have

$$l_{Z_n x_0}(\alpha) \geqslant C_\mu i(\mu, \alpha)\, e^{(l-\epsilon)n}.$$

On the other hand, we clearly also have the upper bound since, for all sufficiently large n,

$$L(x_0, Z_n x_0) = \log \sup_{\alpha} \frac{l_{Z_n x_0}(\alpha)}{l_{x_0}(\alpha)} \leqslant (l+\epsilon)n.$$

In view of these two inequalities, we have, for every $\alpha \in \mathcal{S}$ with $i(\mu, \alpha) > 0$, that

$$l_{Z_n x_0}(\alpha)^{1/n} \to \lambda := e^l.$$

Note that $l_{x_0}(f_n \alpha) = l_{Z_n x_0}(\alpha)$ and that, since M is compact from the point of view of exponential growth, every Riemannian metric is equivalent, so we can replace x_0 with an arbitrary metric ρ. This completes the proof of Theorems 2 and 3. We also remark is that if we do not assume ergodicity, by ergodic decomposition, there is no change other than that the 'Lyapunov exponent' λ is now random and not necessarily constant. A final comment is that M could be allowed to have punctures, except for restricting to Riemannian metrics equivalent to hyperbolic metrics.

2.1. *Additional arguments in the random walk case*

Assume that ν is a probability measure on MCG of finite first moment (here meaning that the cocycle is integrable) and whose support generates a non-elementary subgroup. A non-elementary subgroup is by definition a subgroup which does not fix any finite set in $\mathcal{PMF}$. It is shown in [**13**] that this is equivalent to that there are at least two pseudo-Anosovs with disjoint fixed point sets. In particular it implies that the subgroup is non-amenable. Kaimanovich and Masur [**13**] established that random walks a.s. converge to points in the set of uniquely ergodic foliations of $\mathcal{PMF}$. Moreover, they show that the measure that this convergence gives rise to, the harmonic or hitting measure, is the unique ν-stationary measure. In my work with Ledrappier, we construct special ν-stationary measures in a general setting, like the measure m above; see [**17**, Theorem 18]. By uniqueness, this measure m is hence the same as the hitting one and we get the same conclusion (3) as above (cf. also the last paragraph of the proof of [**17**, Theorem 18]). Now the random foliations μ are uniquely ergodic, so $i(\mu, \alpha) > 0$ for any $\alpha \in \mathcal{S}$. Finally, $\lambda > 1$ because $l > 0$, which in turn is a consequence of entropy theory, including a well-known inequality of Guivarc'h, and the fact that non-elementary subgroup is non-amenable as well as the fact that $\mathcal{T}$ has at most exponential growth as established in [**13**]; see, for example, [**12**] for this type of arguments. This proves Corollary 4.

3. *Proof of Theorem* 5

Recall Kerckhoff's formula for the Teichmüller distance:

$$d(x, y) = \frac{1}{2} \log \sup_{\alpha \in \mathcal{S}} \frac{\mathrm{Ext}_x(\alpha)}{\mathrm{Ext}_y(\alpha)},$$

where Ext denotes extremal length and is defined in the introduction. Royden showed that this metric coincides with the Kobayashi metric associated to the complex structure of $\mathcal{T}$. This implies that holomorophic maps are 1-Lipschitz in this metric; see [**6**] for an update on finer contraction properties. For a comparison between Teichmüller's and Thurston's metrics, see [**4**]. Gardiner and Masur introduced in [**11**] a compactification analogous to the Thurston compactification but using extremal lengths instead of hyperbolic lengths.

Fix a point $x_0 \in \mathcal{T}$. Let

$$E_x(\alpha) = \frac{\mathrm{Ext}_x(\alpha)^{1/2}}{K_x^{1/2}},$$

where K_x is the quasi-conformal dilatation of the Teichmüller map from x_0 to x. Miyachi [23] noted that E extends continuously to a function defined on the Gardiner–Masur compactification $\bar{\mathcal{T}}^{\mathrm{GM}}$of $\mathcal{T}$.

Let $f : \mathcal{T} \to \mathcal{T}$ be a holomorphic self-map of Teichmüller space. Define

$$l = \lim_{n\to\infty} \frac{1}{n} d(f^n x_0, x_0),$$

which exists by the 1-Lipschitz property and subadditivity. Clearly, $0 \leqslant l < \infty$. Moreover, for any point $P \in \bar{\mathcal{T}}^{\mathrm{GM}}$ define, following Liu and Su,

$$h_P(x) = \log \sup_\beta \frac{E_P(\beta)}{\mathrm{Ext}_x(\beta)^{1/2}} - \log \sup_\alpha \frac{E_P(\alpha)}{\mathrm{Ext}_{x_0}(\alpha)^{1/2}}.$$

We set for short

$$Q(P) = \sup_\alpha \frac{E_P(\alpha)}{\mathrm{Ext}_{x_0}(\alpha)^{1/2}}.$$

Given a sequence $\epsilon_i \searrow 0$, we set $b_i(n) = d(f^n x_0, x_0) - (l - \epsilon_i)n$. Since these numbers are unbounded, we can find a subsequence such that $b_i(n_i) > b_i(m)$ for any $m < n_i$ and by sequential compactness we may, moreover, assume that $f^{n_i}(x_0) \to P \in \bar{\mathcal{T}}^{\mathrm{GM}}$; cf. [14].

By a result of Liu and Su [19] identifying the horoboundary compactification of $(\mathcal{T}, d)$ with the Gardiner–Masur compactification (in particular, showing that the latter is metrizable) we have, for any $k \geqslant 1$, that

$$\begin{aligned} h_P(f^k x_0) &= \lim_{i\to\infty} d(f^k x_0, f^{n_i} x_0) - d(x_0, f^{n_i} x_0) \\ &\leqslant \liminf_{i\to\infty} d(x_0, f^{n_i - k} x_0) - d(x_0, f^{n_i} x_0) \\ &\leqslant \liminf_{i\to\infty} b_i(n_i - k) + (l - \epsilon_i)(n_i - k) - b_i(n_i) - (l - \epsilon_i) n_i \\ &\leqslant \liminf_{i\to\infty} -(l - \epsilon_i)k = -lk. \end{aligned}$$

This means, in terms of extremal lengths, that

$$\left(\sup_\beta \frac{E_P(\beta)}{\mathrm{Ext}_{f^k x_0}(\beta)^{1/2}} \right)^{-1} \geqslant Q(P)^{-1}\, e^{lk},$$

and hence, for any $\beta \in \mathcal{S}$, that

$$\mathrm{Ext}_{f^k x_0}(\beta) \geqslant E_P(\beta)^2 Q(P)^{-2}\, e^{2lk}.$$

On the other hand, in view of Kerckhoff's formula one has an estimate from above:

$$e^{2d_T(f^k x_0, x_0)} = \sup_\alpha \frac{\mathrm{Ext}_{f^k x_0}(\alpha)}{\mathrm{Ext}_{x_0}(\alpha)} \geqslant \frac{\mathrm{Ext}_{f^k x_0}(\beta)}{\mathrm{Ext}_{x_0}(\beta)}.$$

In particular, provided $E_P(\beta) > 0$, the two estimates imply that

$$\mathrm{Ext}_{f^n x_0}(\beta)^{1/n} \to e^{2l},$$

which, by letting $\lambda = e^{2l}$ and setting $x = x_0$, completes the proof of Theorem 5.

Note here that the l, or λ, are independent of x because f is 1-Lipschitz, while P might a priori depend on the point x.

3.1. *Additional arguments concluding the proof of Theorem* 6

Suppose that the orbit is unbounded, then by a theorem of Calka, the orbit tends to infinity, see [15]. Moreover, l is defined as above and we suppose that P is a uniquely ergodic point. From Miyachi [23, Proposition 5.1], it then follows that we have that this subsequence also converges to P in the Thurston boundary. In view of [15, Theorem 11 and Corollary 45] it then follows that the whole forward orbit must converge to P (and hence in both compactifications).

In the notation above, we finally have

$$\begin{aligned} h_P(f(y)) &= \lim_{i\to\infty} d(f(y), f^{n_i}x_0) - d(x_0, f^{n_i}x_0) \\ &\leqslant \liminf_{i\to\infty} d(y, f^{n_i-1}x_0) - d(x_0, f^{n_i}x_0) \\ &\leqslant \liminf_{i\to\infty} d(y, f^{n_i-1}x_0) - d(x_0, f^{n_i-1}x_0) - (l-\epsilon_i) \\ &= h_P(y) - l. \end{aligned}$$

From this. we may conclude Theorem 6 as in the previous proof (note that this time we have $\log Q(P)$ on both sides, hence this cancels out).

In the bounded orbit case, we recall that if f is a mapping class, then there is a fixed point by a theorem of Nielsen, if f is a strict contraction, then, by a simple argument due to Edelstein, the orbit converges towards the unique fixed point. However, in general we do not know this.

We end by remarking that [23, Proposition 5.1] also in the reducible case (that is, unbounded orbits but P not uniquely ergodic) gives information on the possible boundary limit points of the orbit in terms of intersections with P.

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