STRONG LAW OF LARGE NUMBERS WITH CONCAVE MOMENTS

ANDERS KARLSSON AND NICOLAS MONOD

Abstract. It is observed that a wellnigh trivial application of the ergodic theorem from [3] yields a strong LLN for arbitrary concave moments.

Not for publication: we found that Aaronson–Weiss essentially proved Theorem 1, see J. Aaronson, *An introduction to infinite ergodic theory* (AMS Math. Surv. Mon. 50, 1997), pages 65–66.

1. Introduction

Let Ω be a standard probability space and $L : \Omega \to \Omega$ and ergodic measure-preserving transformation. Let $f : \Omega \to \mathbf{R}$ be any measurable map and consider the *Birkhoff sums* $S_n = \sum_{k=0}^{n-1} f \circ L^k$. Recall that this ergodic (or "stationary") setting includes the case of the sums $\sum_{k=0}^{n-1} X_k$ of a family $\{X_k\}_k$ of i.i.d. random variables.

Theorem 1. *Let $D : \mathbf{R}_+ \to \mathbf{R}_+$ be any concave function with $D'(\infty) = 0$. If $D(|f|)$ is integrable, then*

$$\lim_{n\to\infty} \frac{1}{n} D(|S_n|) \;=\; 0 \qquad a.s.$$

Remark 2. (i) For the notation $D'(\infty)$, recall that the derivative D' exists except possibly on a countable set and is non-increasing. Moreover, $\lim_{t\to\infty} D(t)/t = D'(\infty)$.
(ii) The condition $D'(\infty) = 0$ is not a restriction, since otherwise straightforward estimates reduce the question to Birkhoff's theorem and $D(|S_n|)/n$ tends to $D'(\infty)|\int f|$.

Playing with the choice of the arbitrary concave function D, one gets examples old and new. For instance, $D(t) = t^p$ yields a Marcinkiewicz–Zygmund theorem:

Corollary 3. *Let $0 < p < 1$. If $f \in L^p$, then $\lim_{n\to\infty} \frac{1}{n^{1/p}} S_n = 0$.* □

Remark 4. Marcinkiewicz–Zygmund [5, §6] work in the i.i.d. case but with $0 < p < 2$; however, when $p > 1$, no such statement can hold in the ergodic generality even under the strongest assumptions, see Proposition 2.2 in [1]. The independence condition was removed by S. Sawyer [8, p. 165]. The beautiful geometric proof by Ledrappier–Lim [4] for $p = 1/2$ has inspired the present note.

Recall that f is *log-integrable* if $\log^+ |f| \in L^1$, where $\log^+ = \max(\log, 0)$. The choice $D(t) = \log(1 + t)$ yields:

Corollary 5. *If f is log-integrable, then $\lim_{n\to\infty} |S_n|^{1/n} = 1$.* □

Observe that for functions $\mathbf{R}_+ \to \mathbf{R}_+$, concavity is preserved under composition. Combining this operation and shift of variables, one has further examples such as:

Corollary 6. *Let $p > 0$. If $\log^+ |f| \in L^p$, then $\lim_{n\to\infty} |S_n|^{n^{-1/p}} = 1$.*

2. Proofs

Theorem 1. Notice that D is non-decreasing and subadditive. We can assume $D(0) = 0$ upon adding a constant. We can assume that D tends to infinity since otherwise it is bounded. Thus, $D(|x - y|)$ defines a proper invariant metric on the group $\mathbf{R}$ and S_n is the random walk associated to (Ω, L). Recall that a *horofunction* h (normalized by $h(0) = 0$) is any limit point in the topology of compact convergence of the family $D(|t - x|) - D(|t|)$ of functions of x indexed by t. According to [3], there are horofunctions h_ω on $\mathbf{R}$ such that

$$\lim_{n\to\infty} \frac{1}{n} D(S_n(\omega)) \;=\; -\lim_{n\to\infty} \frac{1}{n} h_\omega(S_n(\omega)) \qquad \text{a.s.}(\omega)$$

However, $D'(\infty) = 0$ implies that $h = 0$ is the only horofunction since

$$\lim_{t\to\pm\infty} \big(D(|t - x|) - D(|t|)\big) \;= 0 \qquad \forall\, x.$$

□

Corollary 5. For any $p > 0$ there is x_0 such that $D(t) = (\log(x + x_0))^p$ is concave on $\mathbf{R}_+$. Now $D(|f|)$ is integrable and the statement follows. □

3. Comments and references

(i) We used only a very special case of the LLN of [3], which applies to group-valued random variables. Theorem 1 holds indeed also in that setting with identical proof, but can immediately be reduced to the real-valued case.

(ii) The point of the present note is that the LLN of [3] brings new insights even when the group is the range $\mathbf{R}$ of classical random variables, since we can endow it with various invariant metrics. There is indeed a wealth of such metrics; recall that even $\mathbf{Z}$ admits an invariant metric whose completion is Urysohn's universal polish space [9] (by Theorem 4 in [2]). Wild *proper* metrics can be constructed by means of weighted infinite generating sets.

(iii) One can relax the concavity assumption is various ways. For instance, keeping $D(t)/t \to 0$, it suffices to assume that D is quasi-concave in the sense that Jensen's inequalities hold up to a multiplicative constant. Indeed, this implies that D can be constrained within two proportional concave functions [6, Theorem 1].

(iv) V. Petrov [7] discusses laws of the form $S_n/a_n \to 0$ in the i.i.d. case.

References

[1] J. Baxter, R. Jones, M. Lin, and J. Olsen. SLLN for weighted independent identically distributed random variables. *J. Theoret. Probab.*, 17(1):165–181, 2004.

[2] P. J. Cameron and A. M. Vershik. Some isometry groups of the Urysohn space. *Ann. Pure Appl. Logic*, 143(1-3):70–78, 2006.

[3] A. Karlsson and F. Ledrappier. On laws of large numbers for random walks. *Ann. Probab.*, 34(5):1693–1706, 2006.

[4] F. Ledrappier and S. Lim. A proof of a $L^{1/2}$ ergodic theorem. preprint.

[5] J. Marcinkiewicz and A. Zygmund. Sur les fonctions indépendantes. *Fundam. Math.*, 29:60–90, 1937.

[6] H. P. Mulholland. The generalization of certain inequality theorems involving powers. *Proc. Lond. Math. Soc., II. Ser.*, 33:481–516, 1932.

[7] V. V. Petrov. On the strong law of large numbers. *Statist. Probab. Lett.*, 26(4):377–380, 1996.

[8] S. A. Sawyer. Maximal inequalities of weak type. *Ann. of Math. (2)*, 84:157–174, 1966.

[9] P. Urysohn. Sur un espace métrique universel (rédigé par P. Alexandroff). *Bulletin Sc. Math. (2)*, 51:43–64 & 74–90, 1927.

Royal Institute of Technology, Sweden / EPFL and Geneva, Switzerland
E-mail address: akarl@math.kth.se / nicolas.monod@epfl.ch