Spectral zeta functions

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July 10, 2019

Abstract

This paper discusses the simplest examples of spectral zeta functions, especially those associated with graphs, a subject which has not been much studied. The analogy and the similar structure of these functions, such as their parallel definition in terms of the heat kernel and their functional equations, are emphasized. Another theme is to point out various contexts in which these non-classical zeta functions appear. This includes Eisenstein series, the Langlands program, Verlinde formulas, Riemann hypotheses, Catalan numbers, Dedekind sums, and hypergeometric functions. Several open-ended problems are suggested with the hope of stimulating further research.

1 Introduction

Euler observed the following product formula:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}$$

where the product is taken over the prime numbers. This function of the complex variable s is called the Riemann zeta function, denoted $\zeta(s)$, and the expressions above are convergent for $\mathrm{Re}(s) > 1$. The right hand side inspired several generalizations, by Artin, Hasse, Weil, Selberg, Ihara, Artin, Mazur, Ruelle, and others, see [IK04, Te10] and references therein. The most far-reaching frameworks for Euler products might be provided by the insights of Grothendieck and Langlands. The left hand side found generalizations by Dirichlet, Dedekind, Hurwitz, Epstein and others in number theory, but also in another direction in the work of

*Supported in part by the Swiss NSF.

Carleman [Ca34], later extended in [MP49], where instead of the integers one takes Laplacian eigenvalues:

$$\sum_{\lambda \neq 0} \frac{1}{\lambda^s}$$

convergent in some right half-plane. The purpose of the present note is to survey a few recent investigations of these latter functions, *spectral zeta functions*, in cases where the Laplacian is less classical, instead coming from graphs or p-adics. As in one of Riemann's arguments, to get the analytic continuation one rather defines the zeta function via the heat kernel, so the Mellin transform of the heat trace (removing the constant term whenever necessary) and divide by $\Gamma(s)$. This has the advantage of also making sense when the spectrum has continuous part.

One intriguing aspect that we mention is the functional equation of the type s vs $1-s$ that appears also for these non-classical zeta functions. Notably one has

$$\xi_{\mathbb{Z}}(1-s) = \xi_{\mathbb{Z}}(s),$$

see below, and the equivalence of certain asymptotic functional equations to the Riemann hypotheses for $\zeta(s)$ and certain Dirichlet L-functions [FK17, F16].

Although there are a few instances in the literature where such function are introduced for graphs, it seems that the first more systematic effort to study spectral zeta functions of graphs appear in my paper with Friedli [FK17]. As it turns out, via asymptotic considerations, these functions are intimately related to certain zeta functions from number theory. In what follows we will moreover point out that spectral zeta functions for graphs appear *incognito* in, or are connected to, a rich set of topics:

- Eisenstein series, continuous spectrum of surfaces, Langlands program,
- Riemann hypotheses
- Dedekind sums and Verlinde formulas,
- hypergeometric functions of Appell and Lauricella
- Catalan numbers
- Fuglede-Kadison determinants

Although each of these connections may not be of central importance for the corresponding topic, still, in view of the variety of such appearances, I think that the subject of graph spectral zeta functions deserves more attention. This note can also be viewed as a modest update to parts of the discussion in Jorgenson-Lang [JL01] originally entitled *Heat kernels all over the place.* Throughout the text, I will suggest some questions and further directions for research.

I would like the thank the organizers, Mugnolo, Atay and Kurasov, for the very stimulating workshop *Discrete and continuous models in the theory of networks,* at the ZIF, Bielefeld, 2017, and for their invitation. I thank Fabien Friedli, Mårten Nilsson and the referee for corrections.

2 Continuous case

2.1 $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$

This case has been recorded in so many places and there is no need to repeat it here. But for comparison with the other contexts, we recall the formulas. The heat kernel on $\mathbb{R}$ is

$$K_{\mathbb{R}}(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/t}.$$

The heat kernel on the circle $\mathbb{R}/\mathbb{Z}$ is

$$K_{\mathbb{R}/\mathbb{Z}}(t,x) = \sum_{n\in\mathbb{Z}} e^{-4\pi^2 n^2 t} e^{2\pi i n x}.$$

This expression comes from spectral considerations and equals the following periodization (i.e. one sums over the discrete group $\mathbb{Z}$ to obtain a function on the quotient $\mathbb{R}/\mathbb{Z}$, like in the proof of the Poisson summation formula) :

$$\frac{1}{\sqrt{4\pi t}} \sum_{m\in\mathbb{Z}} e^{-(x+n)^2/4t}.$$

Setting $x = 0$ in the former expression, removing 1, and taking the Mellin transform and dividing by $\Gamma(s)$ yields

$$\zeta_{\mathbb{R}/\mathbb{Z}}(s) := 2 \cdot 4^{-s} \pi^{-2s} \zeta(2s),$$

which is the spectral zeta function of the circle $\mathbb{R}/\mathbb{Z}$. If we replace s by $s/2$ we have

$$2 \cdot (2\pi)^{-s} \zeta(s).$$

By defining the completed zeta function (in particular by multiplying back the gamma factor)

$$\xi(s) := \frac{1}{2} 2^s \pi^{s/2} \Gamma(s/2) \zeta_{\mathbb{R}/\mathbb{Z}}(s/2)$$

and equating the two heat kernel expressions (Poisson summation formula) one gets the fundamental functional equation in this form

$$\xi(1-s) = \xi(s),$$

known in the physics literature as the reflection formula for $\zeta(s)$.

The identity with the two heat kernel expressions are known to contain moreover a wealth of theorems, such as the modularity of theta functions and the law of quadratic reciprocity.

2.2 A few comments on further examples

As recalled in the introduction the definition of spectral zeta function was perhaps first in [Ca34], in fact he had more general functions using also the eigenfunctions. This was used for the study of asymptotic properties of the spectrum using techniques from analytic number theory. Another application is for the purpose of defining determinants of Laplacians, in topology this was done by Seeley, Ray and Singer in the definition of analytic torsion, see [R97], and in physics by Dowker, Critchley, and Hawking [DC76, Ha77], useful in several contexts, see [E12]. A referee suggested that in this context one can also make reference to the important heat kernel approach to index theorems, see [BGV92], and to the connection with Arakelov geometry, see [So92].

The determinant of the Laplacian is defined in the following way, assuming the analytic continuation of the zeta function to $s = 0$,

$$\det \Delta_X := e^{-\zeta'_X(0)}.$$

For the numbers $1, 2, 3, ...$ (essentially the circle spectrum) the corresponding determinant, which formally would be $1 \cdot 2 \cdot 3 ... = \infty!$, has the value $\sqrt{2\pi}$ thanks to the corresponding well-known special value of $\zeta(s)$, and this value is coherent with the asymptotics of the factorial function $n!$ by de Moivre and Stirling. In fact it was exactly this constant that Stirling determined.

For tori the corresponding spectral zeta function are Epstein zeta functions. There are also some studies of spheres with explicit formulas. Osgood, Phillips, Sarnak [OPS88] showed that the determinant of the Laplacian is a proper function on the moduli spaces of Riemann surfaces, and could conclude that the set of isospectral surfaces is pre-compact. In string theory it was important to study how the determinant of laplacian changes when varying the metric on surfaces.

Spectral functions of Riemannian manifolds will appear in the limit of spectral zeta functions for certain sequences of graphs. The first more substantial example, the case of tori, can be found in [FK17].

3 Discrete case

To avoid confusion, let me right away emphasize that I am not considering the Ihara zeta function or any related function. Therefore I do not give references for this zeta function, other than the book by Terras [Te10] and another recent references for the Ihara zeta function of quantum graphs [Sm07].

On the other hand, what is more relevant in our context here, but will not be discussed, are the spectral zeta functions of quantum metric graphs studied by Harrison, Kirsten and Weyand, see [HK11, HWK16, HW18] and references therein. The paper [HW18] even discusses the spectral zeta functions of discrete graphs exactly in our sense.

Other related papers on spectral functions of (especially finite cyclic) graphs are those of Knill [Kn13, Kn18] which contain a wealth of interesting ideas, and some considerations close to topics in [FK17] (we do not however understand his Theorem 1c in [Kn13], the convergence to a non-vanishing function in the critical strip, see also his Theorem 10; for us this limiting function is the Riemann zeta function which of course has zeros.) Knill also defines a function $c(s)$ that coincides with our $\zeta_{\mathbb{Z}}(s)$ perhaps without connecting it to the graph $\mathbb{Z}$. The asymptotics for cyclic graphs considered in [Kn13] and [FK17] were also deduced in [Si04]. The re-proof of Euler's formulas for $\zeta(2n)$ via such asymptotics also appears in [CJK10], but also here there are earlier references, for example [W91], in view of the form of Verlinde's formulas, discussed below.

3.1 $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$

Using the heat kernel on $\mathbb{Z}$ in terms of the I-Bessel function, see for example [KN06] where it is rediscovered, the spectral zeta function of the graph $\mathbb{Z}$ is

$$\zeta_{\mathbb{Z}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-2t} I_0(2t) t^s \frac{dt}{t}, \tag{1}$$

where it converges as it does for $0 < Re(s) < 1/2$. From this definition it is not immediate that it admits a meromorphic continuation and a functional equation very similar to classical zetas. However, the following was shown in [FK17]:

Theorem 1. *For all $s \in \mathbb{C}$ it holds that*

$$\zeta_{\mathbb{Z}}(s) = \frac{1}{4^s\sqrt{\pi}} \frac{\Gamma(1/2-s)}{\Gamma(1-s)}.$$

The function

$$\xi_{\mathbb{Z}}(s) = 2^s \cos(\pi s/2)\zeta_{\mathbb{Z}}(s/2),$$

is entire and satisfies for all $s \in \mathbb{C}$

$$\xi_{\mathbb{Z}}(s) = \xi_{\mathbb{Z}}(1-s).$$

Jérémy Dubout [Du16] observed that in fact one can write this function as follows:

$$\zeta_{\mathbb{Z}}(s) = \binom{-2s}{-s}. \tag{2}$$

This makes a connection to the **Catalan numbers**

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

which are ubiquitous in combinatorics (214 such manifestations are listed in the book by Stanley [St12]).

Problem. Could the other $\zeta_{\mathbb{Z}^d}(-n)$ be thought of as generalizations of the Catalan numbers?

Note also that (2) immediately shows, what is not a priori clear from the definition (1), that at negative integers this zeta function takes rational (indeed integral) values. This is analogous to Riemann zeta function and other Dedekind zeta functions, by theorems of Hecke, Klingen, Siegel, Deligne and Ribet. I refer to [Du16] for a fuller discussion on this topic in the graph setting.

Eisenstein series are functions which appear already in classical number theory as well as in the spectral theory of surfaces with cusps. Our function $\zeta_{\mathbb{Z}}(s)$ can be seen to be an important fudge factor (also called scattering determinant) for the Eisenstein series, this comes from Selberg but he does not realize that it is itself a spectral zeta function and writes just

$\sqrt{\pi}\Gamma(s-1/2)/\Gamma(s)$. We think that $\zeta_{\mathbb{Z}}(s)$ is undeniably present, the question is whether its appearance is incidental or part of a general structure. Evidence for the more structural picture could be that for surfaces $\mathbb{Z}$ appears as the fundamental group of the cusps. This leads to the question whether in higher dimensional Eisenstein series, for example in Langlands' work [La76], other graph spectral zeta functions occur. In this context, the **Langlands program,** we observe that our function $\zeta_{\mathbb{Z}}(s)$ (or products thereof) is essentially the value in the Bhanu-Murty-Gindikin-Karpelevich formula at the archimedian place, see [Bh60, FGKP16, Ch. 8] related to the Harish-Chandra c-function.

Problem. Understand the role played by $\zeta_{\mathbb{Z}}(s)$, and perhaps other spectral zeta functions, in the theory of Eisenstein series and the Langlands program. For example, $\zeta_{\mathbb{Z}}(s)$ and all the p-adic zeta functions in the last section might appear together in the Langlands constant term, or Langlands p-adic Gindikin-Karpelevich formula in [La71].

The spectral zeta function of the finite cyclic graph $\mathbb{Z}/n\mathbb{Z}$ (see e.g. [CJK10, FK17] for details) is

$$\zeta_{\mathbb{Z}/n\mathbb{Z}}(s) = \frac{1}{4^s}\sum_{k=1}^{n-1}\frac{1}{\sin^{2s}(\pi k/n)}.$$

There exists some literature on finite trigonometric sums, which in our context appears as the special values

$$\zeta_{\mathbb{Z}/n\mathbb{Z}}(m)$$

for integral m. Some of these special values are recorded in [FK17]. In particular since we can transform $\sin^{-2} z$ to $\cot^2 z$ we have special cases of **Dedekind sums** (which again leads into the theory of Eisenstein series as I learnt from Claire Burrin), see [Z73]. In another context, in the first of the **Verlinde formulas**, such sums appear, see [Sz93, Z96] for mathematical discussions. Let $\mathcal{N}_{\mathrm{g,n,d}}$ denote the moduli space of semi-stable n-dimensional vector bundles over a fixed Riemann surface of genus g and having as determinant bundle a fixed line bundle of degree d. The formula reads

$$\dim_{\mathbb{C}} H^0(\mathcal{N}_{\mathrm{g,2,0}},\mathcal{L}^{\mathrm{m}}) = \frac{(\mathrm{m}+2)^{\mathrm{g}-1}}{2^{\mathrm{g}-1}}\sum_{\mathrm{k}=1}^{\mathrm{m}+1}\frac{1}{\sin^{2\mathrm{g}-2}\frac{\pi \mathrm{k}}{\mathrm{m}+2}}.$$

The right hand side can in our terminology be written as

$$(m+2)^{g-1}2^{g-1}\zeta_{\mathbb{Z}/(m+2)\mathbb{Z}}(g-1).$$

The lead term in the asymptotics as $m \to \infty$ was considered by Witten [W91] to evaluate the volume of the moduli space in question and here the well-known special values $\zeta(2(g-1))$ of the Riemann zeta function appears. This is consistent with the results in [FK17] and indeed earlier in a numerical analysis paper [Si04].

Problem. Is the appearance of the special values of spectral zeta function of cyclic graphs just a coincidence, or are there other cases of the Verlinde formulas that allow interpretations as spectral zeta functions of graphs? If so, this would be intriguing and demand for an explanation.

In [FK17], using some of the methods in [CJK10, CJK12], we study the asymptotics of the spectral zeta function of $\mathbb{Z}^{\mathrm{d}}/A_n\mathbb{Z}^d$ as $n \to \infty$ motivated in particular by statistical physics. One sees there how in the asymptotics, the spectral zeta functions of the infinite graphs and manifold spectral zeta functions appear. In [FK17] and [F16] relations to analytic number theory are discussed. Notably there are reformulations of the **Riemann hypothesis** and the generalized Riemann hypothesis for certain Dirichlet L-functions. For example in [F16]: Let $m \geq 3$ and let χ be a primitive and even Dirichlet character modulo m. For $n \geq 1$, define the cyclic graph L-function

$$L_n(s,\chi) := \sum_{k=1}^{mn-1} \frac{\chi(k)}{\sin^s(\pi k/mn)}.$$

Let $\Lambda_n(s,\chi) = n^{-s}(\pi/k)^{s/2}\Gamma(s/2)L_n(s,\chi)$. Recall the classical Dirichlet L-function

$$L(s,\chi) = \sum_{k=1}^{\infty} \frac{\chi(k)}{k^s}.$$

Friedli's theorem relates the Riemann hypothesis, on the location of the zeros of this latter function to a functional relation for the graph functions that imitates the well-known one for L itself. (This extension of [FK17] seems to me rather surprising.)

Theorem 2. *[F16] The following two statements are equivalent:*

1. *For all s with $0 < Re(s) < 1$ and $Im(s) \geq 8$ we have*

$$\lim_{n\to\infty} \frac{|\Lambda_n(s,\chi)|}{|\Lambda_n(1-s,\overline{\chi})|} = 1;$$

2. *In the region $0 < Re(s) < 1$ and $Im(s) \geq 8$, all zeros of $L(s,\chi)$ have real part $1/2$.*

The first statement holds in any case for all s where L does not vanish. The appearance of a restriction on the imaginary part has a substantial reason, see Lemma 3.1 in the proof of the theorem in [F16]. A related lemma with a similar restriction was proved in the case of the Riemann zeta function and the restriction was shown to be essential [FK17].

3.2 A few comments on further examples

As it is pointed out in [FK17] the spectral zeta functions of such fundamental infinite graphs as the regular trees T_{q+1} and the standard lattices $\mathbb{Z}^d$ lead into **hypergeometric functions** of several variables, more precisely, specializations of these functions.
The spectral zeta function of $\mathbb{Z}^d$ is

$$\zeta_{\mathbb{Z}^d}(s) = \frac{d^{-s+1/2}}{\sqrt{2\pi}} \frac{\Gamma((s+1)/2)}{\Gamma(s)} F_C^{(d)}(s/2, (s+1)/2; 1, 1, ..., 1; 1/d^2, 1/d^2, ..., 1/d^2),$$

where $F_C^{(d)}$ is one of the Lauricella hypergeometric functions in d variables.

Problem. It is remarked in [Ex76, p. 49] that no integral representation of Euler type has been found for F_C. We note that if one instead of the heat kernel starts with the spectral measure in defining $\zeta_{\mathbb{Z}^d}(s)$, we do get such an integral representation, at least for special parameters. Does this lead to the missing Euler-type integral representation formula?

The $(q+1)$-regular tree (or Bethe lattice in physics parlance) is the universal covering of $(q+1)$-regular graphs and is therefore a fundamental graph. In [FK17] an expression for the corresponding zeta function is found, interestingly via an Euler-type integral that Picard considered and which leads into Appell's hypergeometric function F_1:

$$\zeta_{T_{q+1}}(s) = \frac{q(q+1)}{(q-1)^2(\sqrt{q}-1)^{2s}} F_1(3/2, s+1, 1, 3; u, v),$$

with $u = -4\sqrt{q}/(\sqrt{q}-1)^2$ and $v = 4\sqrt{q}/(\sqrt{q}+1)^2$.

Problem. What functional equations do these spectral zeta functions have? In view of the many symmetries that such hypergeometric functions have, could one hope for an s vs $1-s$ symmetry or similar identities?

The determinant of the Laplacian (of course removing the trivial eigenvalue 0 from the product) of a finite graph is known to count the number of spanning trees (with a root) of the graph. This is called Kirchhoff's matrix-tree theorem. For infinite graphs the corresponding determinant is sometimes related to Mahler measures (in number theory) and more generally to **Fuglede-Kadison determinants** (in operator algebras). See [Ly10, CJK10, CJK12] for more about these connections. A famous value is the determinant of the graph $\mathbb{Z}^2$ which is $4/\pi$ times Catalan's constant; one my wonder if there are other such values in terms of special values of Dirichlet L-functions.

4 Totally disconnected case

4.1 $\mathbb{Q}_p$ and $\mathbb{Q}_p/\mathbb{Z}_p$

This follows [CZ17] and the recent master thesis of Mårten Nilsson [Ni18], see these two sources for further references. Let $\mathbb{Q}_p$ denote the p-adic numbers and $\mathbb{Z}_p$ the p-adic integers. Let dy be the Haar measure on the locally compact additive group $\mathbb{Q}_p$ normalized so that the measure of $\mathbb{Z}_p$ is 1. The Taibleson-Vladimirov Laplcaian can be defined like pseudo-differential operators via the Fourier transform, alternatively it is explicitly given as an integral as follows:

$$\Delta f(x) = \frac{p^2-1}{1-p^{-3}} \int_{\mathbb{Q}_p} \frac{f(x)-f(y)}{|x-y|_p^3} dy$$

for suitable functions $f : \mathbb{Q}_p \to \mathbb{C}$. This gives rise in the usual ways to a heat equation and a heat kernel, which in this case turns out to be:

$$K_p(x,t) = \sum_{k=-\infty}^{\infty} \left(e^{-tp^{2k}} - e^{-tp^{2k+2}} \right) p^k C_{p^{-k}}(x),$$

where C_{p^n} denotes the characteristic function of the the ball $B_{p^n} = \left\{ x \in \mathbb{Q}_p : |x|_p \leq p^n \right\}$. There is another formula for this function:

$$K_p(x,t) = \sum_{m=0}^{\infty} \frac{(-1)^m t^m}{m!} \frac{1-p^{2m}}{1-p^{-2m-1}} |x|_p^{-2m-1},$$

valid for $x \neq 0$. Now passing to the quotient $\mathbb{Q}_p/\mathbb{Z}_p$ (which is analogous to the periodization done above and is here an integral over $\mathbb{Z}_p$), and after that taking the Mellin transform dividing by $\Gamma(s)$, leads to the corresponding spectral zeta function which is

$$\zeta_p(s) = (1-p^{-1}) \frac{p^{1-2s}}{1-p^{1-2s}} = \frac{p^{1-2s}-p^{-2s}}{1-p^{1-2s}} = \frac{p-1}{p^{2s}-p}.$$

Here we complete this function in the following manner:

$$\xi_p(s) = \sin(2\pi s)p^s\zeta_p(s),$$

then we obtain a functional equation of the usual type in the most symmetric form:

$$\xi_p(1-s) = \xi_p(s).$$

Since the zeta functions became so simple this relation is more trivial than what we saw in the other contexts. But still, we see a pattern: From the line $\mathbb{R}$ and the circle $\mathbb{R}/\mathbb{Z}$, to the graphs $\mathbb{Z}$ and $\mathbb{Z}/\mathrm{n}\mathbb{Z}$, and now $\mathbb{Q}_p$ and $\mathbb{Q}_p/\mathbb{Z}_p$, their associated spectral zeta functions have a non-obvious symmetry s vs $1-s$.

In another direction taking the Laplace transform of the two heat kernel expressions, Nilsson derives an identity valid for $Re(s) > (p/\left|x\right|_p)^2$:

$$\frac{1}{s\left|x\right|_p}\sum_{m=0}^{\infty}\frac{(-1)^m}{s^m\left|x\right|_p^{2m}}\frac{1-p^{2m}}{1-p^{-2m-1}} = \sum_{k=-\infty}^{\infty}\frac{p^2-1}{(1+p^{2k}s)(p^2+p^{2k}s)}p^kC_{p^k}(x).$$

Specializing to certain x gives an identity without p-adics, but only involving ordinary integers. For example, with $x = 1$ and $s > p^2$,

$$\frac{1}{s}\sum_{m=0}^{\infty}\frac{(-1)^m}{s^m}\frac{1-p^{2m}}{1-p^{-2m-1}} = \sum_{k=0}^{\infty}\frac{p^2-1}{(1+p^{2k}s)(p^2+p^{2k}s)}\cdot p^k.$$

It is natural to compare these considerations with the celebrated **Tate's thesis**, where Tate in particular after a Mellin transform, obtains for each p the local Euler factor $(1-p^{-s})^{-1}$, instead of our $\zeta_p(s)$. This brings us back to the first paragraph of this paper.

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