VOLUMES OF SPHERES AND SPECIAL VALUES OF ZETA FUNCTIONS OF $\mathbb{Z}$ AND $\mathbb{Z}/n\mathbb{Z}$

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ABSTRACT. The volume of the unit sphere in every dimension is given an interpretation as a product of special values of the zeta function of $\mathbb{Z}$, akin to volume formulas of Minkowski and Siegel in the theory of arithmetic groups. A product formula is found for this zeta function that specializes to Catalan numbers. Moreover, certain closed-form expressions for various other zeta values are deduced, in particular leading to an alternative perspective on Euler's values for the Riemann zeta function.

1. INTRODUCTION

The determination of circumference, area, and volume of spheres is one of the oldest topics in geometry. A more sophisticated volume formula, due to Minkowski, is the following:

$$\mathrm{vol}(\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})) = \zeta(2)\zeta(3)...\zeta(n),$$

where $\zeta(s)$ is the Riemann zeta function, and with a suitable coherent choice of normalization of the Haar measure. This is not an incidental fact, instead it is part of a more general phenomenon discovered and developed by Siegel, Weil, Langlands, Harder, and others. It takes the form

$$\mathrm{vol}(G/\Gamma) = c^{[K:\mathbb{Q}]} \prod_{i=1}^{l} \zeta_K(-m_i),$$

and without going into details about this formula and when it holds (see [H71, P01] for more information), let us just highlight one further example:

$$\mathrm{vol}(\mathrm{Sp}_n(\mathbb{R})/\mathrm{Sp}_n(\mathbb{Z})) = \zeta(2)\zeta(4)...\zeta(2n).$$

The passage from negative integers to positive ones is done by means of the fundamental functional equation. In the present paper we will in particular provide a zeta value interpretation of the $(n-1)$-dimensional volume of spheres, which are the homogeneous spaces $S^{n-1} \cong \mathrm{SO}(n)/\mathrm{SO}(n-1)$.

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As is well known, the Riemann zeta function is essentially the spectral zeta function of the circle $\mathbb{R}/\mathbb{Z}$, more precisely

$$\zeta(s)=\frac{1}{2}(2\pi)^s\zeta_{\mathbb{R}/\mathbb{Z}}(s/2)=\frac{(2\pi)^s}{2}\frac{1}{\Gamma(s)}\int_0^\infty\frac{1}{\sqrt{4\pi t}}\sum_{k\neq 0}e^{-k^2/4t}t^{s/2}\frac{dt}{t},$$

for $\mathrm{Re}(s)>1$ and then extended by meromorphic continuation. From Fourier analysis we know that the circle and the integers are dual groups. The function $e^{-2t}I_0(2t)$, with the 0th order I-Bessel function appearing, is the $\mathbb{Z}$-analog of the theta series for the circle inside the Mellin transform expression above. Therefore, entirely analogously to $\mathbb{R}/\mathbb{Z}$ as is explained in [FK17], one can define the spectral zeta function of $\mathbb{Z}$ as

$$\zeta_{\mathbb{Z}}(s)=\frac{1}{\Gamma(s)}\int_0^\infty e^{-2t}I_0(2t)t^s\frac{dt}{t},$$

for $0<\mathrm{Re(s)}<1/2$ and then extend by a meromorphic continuation as will soon become clear. In parallel to the above, we next define the function

$$Z(s)=\frac{1}{2}2\pi 2^s\zeta_{\mathbb{Z}}(s/2),$$

which shares with $\zeta(s)$ the property of having a functional equation of the type of a $s\longleftrightarrow 1-s$ symmetry ([FK17]). We observe the following formula very reminiscent of the above volume expressions:

Theorem 1.1. *For $n>0$, the n-dimensional volume of the unit sphere in $\mathbb{R}^{n+1}$ is*

$$\mathrm{vol}(S^n)=2\cdot Z(0)\cdot Z(-1)...\cdot Z(-n+1).$$

It turns out, by [FK17], and then by Dubout's formula in [D19], that $\zeta_{\mathbb{Z}}(s)$ is an analytic continuation essentially of the Catalan numbers

$$\zeta_{\mathbb{Z}}(s)=\frac{1}{4^s\sqrt{\pi}}\frac{\Gamma(1/2-s)}{\Gamma(1-s)}=\begin{pmatrix}-2s\\-s\end{pmatrix}.$$

The appearance of the gamma function begins explaining the volume formula. Indeed, with the values $Z(0)=\pi$, $Z(-1)=2$ and $Z(-2)=\pi/2$ one sees that the volume formula is true for the first cases. One could alternatively say that the Catalan numbers

$$C_m=\frac{1}{m+1}\begin{pmatrix}2m\\m\end{pmatrix},$$

but now at the nonstandard indices $m=k/2$, appear in the classical expression of the surface area of spheres.

The spectral zeta function of $\mathbb{Z}$ has the following beautiful product formula:

Theorem 1.2. *It holds that*

$$\zeta_{\mathbb{Z}}(s)=\prod_{k=1}^{\infty}\frac{(k-s)^2}{k(k-2s)}$$

interpreted suitably when s is a positive integer or half-integer.

From this expression, the simple zeros and poles of $\zeta_{\mathbb{Z}}(s)$ at the positive integers and half-integers respectively, are clearly visible. As will be shown below, it specializes to the standard product expression for Catalan numbers:

$$C_m=\prod_{k=2}^{m}\frac{m+k}{k}.$$

In the following we shall also provide an exposition of special values of $\zeta_{\mathbb{Z}}(s)$ and $\zeta(s)$, including correcting a few minor inaccuracies in [FK17], as well as studying the related and analogously defined spectral zeta function of $\mathbb{Z}/n\mathbb{Z}$,

$$\zeta_{\mathbb{Z}/n\mathbb{Z}}(s)=\frac{1}{2^{2s}}\sum_{k=1}^{n-1}\frac{1}{\sin^{2s}(\pi k/n)}.$$

The special values $\zeta_{\mathbb{Z}/n\mathbb{Z}}(g)$, their limits as $n\to\infty$, and the values $\zeta(2g)$ all appear in a more elaborate volume context in Witten's paper [W91].

In the present note we will provide an elementary calculation of $\zeta_{\mathbb{Z}/n\mathbb{Z}}(1/2-m)$, and indicate how then to pass to Euler's $\zeta(1-2m)$ values from the asymptotics expansions in [Si04, FK17, MV22]. After that via the symmetry s vs $1-s$ one gets $\zeta(2m)$, which leads to $\zeta_{\mathbb{Z}/n\mathbb{Z}}(m)$.

In this way we connect $\zeta_{\mathbb{Z}/n\mathbb{Z}}(2m/2)$ to $\zeta_{\mathbb{Z}/n\mathbb{Z}}((1-2m)/2)$. Note that asymptotical symmetries of the type s vs $1-s$ for $\zeta_{\mathbb{Z}/n\mathbb{Z}}(s/2)$ and related finite sums is not a trivial matter, in fact certain versions of it are equivalent to various Riemann hypotheses: for $\zeta(s)$ as shown in [FK17], for certain Dirichlet L-functions as proven in [F16], and for the Dedekind zeta function of the Gaussian rationals as established in [MV22].

2. Proof of the volume and product formulas

Proof of the volume formula. The well-known volume (hypersurface area) of spheres is

$$\operatorname{vol}(S^n)=\frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)},$$

generating the numbers $2, 2\pi, 4\pi, 2\pi^2$ etc.

This leads to the following calculation

$$\frac{\operatorname{vol}(S^n)}{\operatorname{vol}(S^{n-1})}=\frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}\frac{\Gamma(n/2)}{2\pi^{n/2}}=\frac{\sqrt{\pi}\Gamma(n/2)}{\Gamma(n/2+1/2)}.$$

This we can rewrite as follows

$$\frac{\mathrm{vol}(S^n)}{\mathrm{vol}(S^{n-1})} = \frac{\sqrt{\pi}\Gamma(1/2-(1/2-n/2)}{\Gamma(1-(1/2-n/2))} = \pi 4^{1/2-n/2}\zeta_{\mathbb{Z}}((1-n)/2) = Z(-n+1).$$

Since $\mathrm{vol}(S^0)=2$ we inductively arrive at the proof of the formula stated in Theorem 1.1.

Proof of the product formula. Recall that the Euler's beta function has the following product formula [AR99, Formula 1.1.26]

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\frac{x+y}{xy}\prod_{k=1}^{\infty}\frac{(1+(x+y)/k)}{(1+x/k)(1+y/k)}$$

with poles for x or y equal to 0 or a negative integer, and analytic elsewhere. On the other hand, Dubout's expression gives

$$\begin{aligned}\zeta_{\mathbb{Z}}(s) &= \frac{\Gamma(1-2s)}{\Gamma(1-s)\Gamma(1-s)} = \frac{\Gamma(2-2s)}{(1-2s)\Gamma(1-s)\Gamma(1-s)}\\ &= \frac{(1-s)^2}{(1-2s)(2-2s)}\prod_{k=1}^{\infty}\frac{(1+(1-s)/k)^2}{(1+(2-2s)/k)}\\ &= \frac{(1-s)^2}{(1-2s)(2-2s)}\prod_{k=1}^{\infty}\frac{(k+1-s)^2}{k(k+2-2s)} = \prod_{k=1}^{\infty}\frac{(k-s)^2}{k(k-2s)},\end{aligned}$$

as was to be proved.

To see that it gives back a correct expression in the case of $s=-m$ a negative integers (the Catalan number case) note that except for the small values, the others integer appear twice in the numerator as well as in the denominator:

$$\begin{aligned}C_m &= \frac{1}{m+1}\zeta_{\mathbb{Z}}(-m) = \frac{1}{m+1}\prod_{k=1}^{\infty}\frac{(k+m)^2}{k(k+2m)}\\ &= \frac{1}{m+1}\frac{(m+1)^2(m+2)^2...}{1\cdot 2...(1+2m)(2+2m)...} = \prod_{k=2}^{m}\frac{m+k}{k},\end{aligned} \tag{2.1}$$

which is an expression whose validity is immediate from the definition of C_m.

3. Special values of $\zeta_{\mathbb{Z}}(s)$ and $\zeta'_{\mathbb{Z}}(s)$

To begin with, note that the special values $\zeta_{\mathbb{Z}}(-m)$, $m\in\mathbb{N}$, are rational, indeed integral. This is not a priori obvious, but in view of the Dubout formula for $\zeta_{\mathbb{Z}}$ this becomes clear since they are just binomial coefficients at these points. This is the counterpart of theorems by Hecke, Siegel, Klingen, and others for classical zeta functions.

The following is taken from [FK17, Pa22]. First, we recall that for integers $n \geq 0$, we have

$$\Gamma(n+1) = n! \quad \text{and} \quad \Gamma\left(\frac{1}{2}+n\right) = \frac{(2n)!}{4^n n!}\pi^{\frac{1}{2}},$$

the latter being the Legendre duplication formula. Thus, by using these formulas and the expressions above, we get for integers $n \geq 1$ that

$$\begin{aligned}
&\zeta_{\mathbb{Z}}(0) = 4^0 \pi^{-\frac{1}{2}} \frac{\Gamma(\frac{1}{2})}{\Gamma(1)} = 1, \\
(3.1) \qquad &\zeta_{\mathbb{Z}}(-n) = 4^n \pi^{-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}+n)}{\Gamma(1+n)} = \frac{(2n)!}{n!n!} = \binom{2n}{n}, \\
&\zeta_{\mathbb{Z}}(-n+\tfrac{1}{2}) = 4^{n-1/2}\pi^{-\frac{1}{2}} \frac{\Gamma(n)}{\Gamma(\frac{1}{2}+n)} = \frac{4^{2n}}{2\pi n}\frac{n!n!}{(2n)!} = \frac{4^{2n}}{2\pi n}\binom{2n}{n}^{-1}.
\end{aligned}$$

As said before the values at positive integers and positive half-integers are precisely the zeros and poles.

Now, if we differentiate $\zeta_{\mathbb{Z}}(s)$, we get

$$\begin{aligned}
\zeta_{\mathbb{Z}}'(s) &= \pi^{-\frac{1}{2}} \left(\frac{\Gamma(\frac{1}{2}-s)}{4^s \Gamma(1-s)} \right)' \\
&= \pi^{-\frac{1}{2}} \frac{-\Gamma(\frac{1}{2}-s)\psi_0(\frac{1}{2}-s) - \Gamma(\frac{1}{2}-s)\big(\log(4) - \psi_0(1-s)\big)}{4^s\Gamma(1-s)} \\
&= \zeta_{\mathbb{Z}}(s)\left(-\psi_0\left(\frac{1}{2}-s\right) - 2\log(2) + \psi_0(1-s)\right),
\end{aligned}$$

since $\Gamma(\frac{1}{2}-s)' = -\Gamma(\frac{1}{2}-s)\psi_0(\frac{1}{2}-s)$ and $(4^s\Gamma(1-s))' = 4^s\Gamma(1-s)(\log(4) - \psi_0(1-s))$, and where $\psi_n(s)$ is the polygamma function, which is defined by

$$\psi_n(s) = \frac{d^n}{ds^n}\frac{\Gamma'(s)}{\Gamma(s)}.$$

We can therefore deduce the following special values of $\zeta_{\mathbb{Z}}'$.

Proposition 3.1. *It holds that*

$$\zeta_{\mathbb{Z}}'(0) = 0 \quad \textit{and} \quad \zeta_{\mathbb{Z}}'(-\tfrac{1}{2}) = \frac{8}{\pi}(1 - 2\log(2)),$$

and

$$\begin{aligned}
\zeta_{\mathbb{Z}}'(-n) &= \binom{2n}{n}\left(\sum_{k=1}^{n} \frac{1}{k} - \frac{2}{2k-1}\right) \\
\zeta_{\mathbb{Z}}'(-n+\tfrac{1}{2}) &= \frac{4^{2n}}{2\pi n}\binom{2n}{n}^{-1}\left(-4\log(2) - \sum_{k=1}^{n-1}\frac{1}{k} + 2\sum_{k=1}^{n}\frac{1}{2k-1}\right)
\end{aligned}$$

for $n \geq 1$ and $n \geq 2$, respectively.

Proof. According to 6.3.2 and 6.3.3 in [AS64], we have the following values

$$\psi_0\left(\frac{1}{2}\right) = -\gamma - 2\log(2) \quad \text{and} \quad \psi_0(1) = -\gamma. \tag{3.2}$$

Furthermore, by formulas 6.3.2 and 6.3.4 in [AS64], we also have

$$\psi_0\left(n+\frac{1}{2}\right) = -\gamma - 2\log(2) + 2\sum_{k=1}^{n} \frac{1}{2k-1} \qquad \text{for } n \geq 1, \tag{3.3}$$

$$\psi_0(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k} \qquad \text{for } n \geq 2. \tag{3.4}$$

Hence, combining (3.2), (3.3), (3.4) and the special values already computed in (3.1) concludes the proof of the proposition. □

Note that these values correct (confirmed by computer numerics) small errors in [FK17, Proposition 6.1]. Indeed, the second formula in that reference should have $-4\log 2$ instead of $-4\log 4$, and the case $n = 1$ needs to be interpreted correctly in view of the term $1/(n-1)$ appearing. Finally it is stated in [FK17] that $\zeta_{\mathbb{Z}}'(s)$ is zero at the positive integers, this is not true (checked by numerics) and instead the values are given here:

Proposition 3.2. *Let n be a positive integer, then*

$$\zeta_{\mathbb{Z}}'(n) = \frac{1}{n}\binom{2n}{n}^{-1}.$$

Proof. By the reflection formula 6.3.7 in [AS64], we have

$$\psi_0(1-z) = \psi_0(z) + \pi\cot(\pi z).$$

Hence, applying this last formula, as well as (3.3) and (3.4), gives

$$\begin{aligned}\zeta_{\mathbb{Z}}'(n) &= \binom{-2n}{-n}\left(-\psi_0\left(\frac{1}{2}-s\right) - 2\log(2) + \psi_0(1-n)\right)\\ &= \binom{-2n}{-n}\left(\sum_{k=1}^{n-1}\frac{1}{k} - 2\sum_{k=1}^{n}\frac{1}{2k-1} + \pi\cot(\pi n)\right).\end{aligned}$$

We observe that for any positive integer n, we have $\binom{-2n}{-n} = 0$. Therefore the two sums are eliminated by the multiplication with the binomial. However we also note that we have $\cot(\pi n) = \pm\infty$. Hence

$$\zeta_{\mathbb{Z}}'(n) = \binom{-2n}{-n}\pi\cot(n\pi) = \frac{\Gamma(1-2n)}{\Gamma(1-n)^2}\pi\cot(n\pi) = \frac{1}{n}\binom{2n}{n}^{-1},$$

where the last equality is obtained first by applying the reflection formula $\Gamma(1-z)\Gamma(z) = \pi\sin^{-1}(\pi z)$, then by applying the recurrence formula $\Gamma(z+1) = z\Gamma(z)$ and finally by using the double-angle formulas for sine. □

4. Special values of $\zeta_{\mathbb{Z}/n\mathbb{Z}}(s)$ and $\zeta(s)$

Sums of powers of the sine function are special cases of important sums in works of Dedekind, Verlinde, Dowker and others in number theory and physics, see [Do92, Z96, Do15, K20]. From our perspective, and also partly from Dowker's, they are special values of spectral zeta functions of discrete circles. It is therefore of interest to recall the following values, see for example [Me14, dF17]. Let n and m be two positive integer, then we have

$$\zeta_{\mathbb{Z}/n\mathbb{Z}}(-m) = 2^{2m}\sum_{k=1}^{n-1}\sin^{2m}\left(\frac{\pi k}{n}\right) = n\sum_{k=-\lfloor\frac{m}{n}\rfloor}^{\lfloor\frac{m}{n}\rfloor}(-1)^{kn}\binom{2m}{m+kn}.$$

In the special case $m < n$, we have

$$\zeta_{\mathbb{Z}/n\mathbb{Z}}(-m) = 2^{2m}\sum_{k=1}^{n-1}\sin^{2m}\left(\frac{\pi k}{n}\right) = n\binom{2m}{m}.$$

Comparing the last formula to the one of $\zeta_{\mathbb{Z}}(-m)$ above (3.1), one sees a manifestation of all the trivial zeros of the Riemann zeta function at the negative even integers, in view of the asymptotics in [Si04, FK17, MV22]

$$\sum_{k=1}^{n-1}\frac{1}{\sin^s(k\pi/n)} = \frac{1}{\sqrt{\pi}}\frac{\Gamma(1/2-s/2)}{\Gamma(1-s/2)}n + 2\pi^{-s}\zeta(s)n^s + \frac{s}{3}\pi^{2-s}\zeta(s-2)n^{s-2} + \ldots$$

as $n \to \infty$. This asymptotic relation demonstrates the intimate connection between the three zeta functions appearing in this paper, $\zeta(s)$, $\zeta_{\mathbb{Z}}(s)$ and $\zeta_{\mathbb{Z}/n\mathbb{Z}}(s)$.

Friedli provided us with the proof of the following formula that was empirically discovered in [Pa22]. Note that as $n \to \infty$ the right hand side remains a sum with m terms:

Proposition 4.1. *Let n and m be positive integer, then*

$$\zeta_{\mathbb{Z}/n\mathbb{Z}}\left(-\frac{1}{2}-m\right) = 2^{2m+1}\sum_{k=1}^{n-1}\sin^{2m+1}\left(\frac{\pi k}{n}\right)$$
$$= 2\sum_{j=0}^{m}(-1)^{m-j}\binom{2m+1}{j}\cot\left(\frac{2m+1-2j}{2n}\pi\right).$$

Proof. First we recall

$$\sin^{2m+1}(k\pi/n) = (2i)^{-2m-1}\left(e^{i\pi k/n} - e^{-i\pi k/n}\right)^{2m+1}$$

$$= (2i)^{-2m-1}\sum_{j=0}^{2m+1}(-1)^j\binom{2m+1}{j}e^{i\pi k(2j-2m-1)/n}.$$

This gives switching the finite sums

$$\sum_{k=1}^{n-1} \sin^{2m+1}(k\pi/n) = (2i)^{-2m-1} \sum_{j=0}^{2m+1} (-1)^j \left(\begin{array}{c} 2m+1 \\ j \end{array} \right) \sum_{k=1}^{n-1} e^{i\pi k(2j-2m-1)/n}.$$

The interior sum is of geometric type and can therefore be summed and equals

$$-i \cdot \cot\left(\pi(2j-2m-1)/2n\right).$$

Observe that there is a symmetry j vs $2m+1-j$

$$\cot\left(\pi(2(2m+1-j)-2m-1)/2n\right) = -\cot\left(\pi(2j-2m-1)/2n\right).$$

Using the same symmetry for the binomial coefficients one arrives at

$$\sum_{k=1}^{n-1} \sin^{2m+1}\left(\frac{\pi k}{n}\right) = \frac{1}{2^{2m}} \sum_{j=0}^{m} (-1)^{m-j} \binom{2m+1}{j} \cot\left(\left(\frac{2m+1-2j}{2n}\right)\pi\right).$$

□

Thanks to the above asymptotics and the well-known series expansion of the cotangent function

$$\cot(z) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} z^{2n-1},$$

one can deduce Euler's celebrated formulas that for odd positive integers m,

$$\zeta(-m) = (-1)^m \frac{B_{m+1}}{m+1},$$

and for even positive m, $zeta(-m) = 0$ (as already remarked above) , together with $\zeta(0) = -1/2$. Indeed, setting $s = 0$ in the asymptotics one has

$$n - 1 = \frac{1}{\sqrt{\pi}} \frac{\Gamma(1/2)}{\Gamma(1)} n + 2\pi^0 \zeta(0) n^0 + ...$$

giving $\zeta(0) = -1/2$. Specializing to $s = -1$ produces

$$\sum_{k=1}^{n-1} \sin(k\pi/n) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(1)}{\Gamma(3/2)} n + 2\pi\zeta(-1) n^{-1} + \frac{-1}{3}\pi^3 \zeta(-3) n^{-3} + ...$$

and on the other hand from Proposition 4.1,

$$\sum_{k=1}^{n-1} \sin(k\pi/n) = \cot(\pi/2n) = \frac{2n}{\pi} B_0 + \frac{\pi}{2n}(-2)B_2 + \frac{\pi^3}{8n^3}\frac{16B_4}{24} + O(1/n^5).$$

This gives that

$$\zeta(-1) = -\frac{B_2}{2} = -\frac{1}{12},\ \zeta(-3) = -\frac{B_4}{4} = \frac{1}{120}, ...$$

In view of the functional equation of $\zeta(s)$ one can then get the values of $\zeta(2m)$ such as $\pi^2/6$, $\pi^4/90$ etc. which appear in the volume formulas in the introduction. This, in turn, again via the above stated asymptotics (the asymptotics for these special s here is just a formula due to the trivial zeros $\zeta(-2m) = 0$) gives the values in closed form of

$$\zeta_{\mathbb{Z}/n\mathbb{Z}}(m) = \frac{1}{4^s} \sum_{k=1}^{n-1} \frac{1}{\sin^{2m}(\pi k/n)}.$$

For example,

$$\zeta_{\mathbb{Z}/n\mathbb{Z}}(1) = 0n + \frac{2}{4\pi^2}\frac{\pi^2}{6}n^2 + \frac{2}{12}\pi^{2-2}\left(-\frac{1}{2}\right) + 0 = \frac{1}{12}(n^2-1)$$

and

$$\zeta_{\mathbb{Z}/n\mathbb{Z}}(2) = \frac{1}{720}\left(n^4 + 10n^2 + 11\right).$$

An easier approach to these last two formulas can be found in [Do92, Z96]. In contrast, the values of $\zeta_{\mathbb{Z}/n\mathbb{Z}}$ at the positive half-integral points do not have such polynomial expressions, which is related to the elusive nature of the zeta values $\zeta(2m+1)$, $m > 0$.

Here we determined $\zeta(-m)$ from the asymptotics of $\zeta_{\mathbb{Z}/n\mathbb{Z}}(-1/2)$, while in the literature one finds in several places the approach from $\zeta_{\mathbb{Z}/n\mathbb{Z}}(g)$ to $\zeta(2g)$, $g > 0$ as $n \to \infty$. One of the more general and rigorous such latter limit formulas can be found in [CJK10, section 7.3].

Let us remark that the Riemann Hypothesis can be reformulated solely in terms of a hypothetical functional symmetry of the standard type $s \longleftrightarrow 1-s$ for $\zeta_{\mathbb{Z}}(s/2)$ and $\zeta_{\mathbb{Z}/n\mathbb{Z}}(s/2)$. This is not an incidental fact, as it was shown to extend in [F16] and [MV22] to some cases of a generalized Riemann Hypothesis.

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