Long time behaviour of the solution to non-linear Kraichnan equations

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28/01/04

Abstract

We consider the solution of a nonlinear Kraichnan equation

$$\partial_s H(s,t) = \int_t^s H(s,u)H(u,t)k(s,u)du$$

with a covariance kernel k and boundary condition $H(t,t) = 1$. We study the long time behaviour of H as the time paramters t,s go to infinity, according to the asymptotic behaviour of k. This question appears in various subjects since it is related with the analysis of the asymptotic behaviour of the trace of non-commutative processes satisfying a linear differential equation, but also naturally shows up in the study of the so-called response function and aging properties of the dynamics of some disordered spin systems.

Keywords : Random matrices, Disordered systems, Large deviations, Statistical mechanics, Langevin dynamics.

Mathematics Subject of Classification : 15A52, 60F10, 60K35, 82C44, 82C31.

1 Introduction

In this paper, we shall consider the long time behaviour of the solution of a nonlinear Kraichnan equation

$$\partial_s H(s,t) = \int_t^s H(s,u)H(u,t)k(s,u)du \tag{1}$$

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with a covariance kernel k and boundary condition $H(t,t)=1$. Such an equation already appeared in the work of Kraichnan [13] as a first term in a perturbative method to analyze quantum-mechanical, turbulence or disordered problems. Shortly afterwards, Frisch and Bourret [12] have shown that these equations naturally appeared when one considers parastochastic equations, which are related with differential equations for non-commutative processes (so-called master equations) and large random matrices. This relation was later studied also by Neu and Speicher [14]. Let us briefly describe it.

Let $(L_t)_{t\geq 0}$ be a process in a von Neumann algebra $\mathcal{A}$ equipped with a tracial state ϕ. We assume that L is a centered semicircular process with covariance kernel k, usually constructed on the full Fock space (see e.g. [15]). In a more intuitive way, L can be constructed as the limit of self-adjoint large random matrices $(L_t^N)_{t\geq 0}$ with entries $\{(L_t^N)_{ij}, 1\leq i\leq j\leq N\}$ which are independent Gaussian processes with covariance $N^{-1}k$. This limit has to be understood in the weak sense that for any integer number n, any times $(t_1,t_2,\cdots t_n)\in(\mathbb{R}^+)^n$,

$$\lim_{N\to\infty}\frac{1}{N}\mathrm{tr}\left(L_{t_1}^N L_{t_2}^N\cdots L_{t_n}^N\right)=\phi\left(L_{t_1}L_{t_2}\cdots L_{t_n}\right).$$

In this paper, we consider the family of operators $\mathbf{X}_{s,t}$ satisfying the linear differential equation

$$\partial_s\mathbf{X}_{s,t}=L_s\mathbf{X}_{s,t},\ s>t,$$

with boundary data $\mathbf{X}_{t,t}=1$, on the Full Fock space. Then, it was shown in [12] that $H(s,t)=\phi(\mathbf{X}_{s,t})$ satisfies Kraichnan's equation (1) (see also section 2 for details). We study the asymptotic behaviour of $H(s,t)=\phi(\mathbf{X}_{s,t})$ as s and t go to infinity. Notice that $\phi(\mathbf{X}_{s,t})$ describes the large N limit of the mean normalized trace $N^{-1}\mathrm{tr}(\mathbf{X}_{s,t}^N)$ of the solution $\mathbf{X}_{s,t}^N$ of the random linear differential equation $\partial_s\mathbf{X}_{s,t}^N=L_s^N\mathbf{X}_{s,t}^N,\ s>t$ such that $\mathbf{X}_{t,t}^N=I$.

Such a question would of course be trivial in the classical setting where L would just be a real-valued Gaussian process. Indeed, in this case, $\mathbf{X}_{t,s}=e^{\int_t^s L_u du}$ and one easily studies the asymptotics of $\phi(\mathbf{X}_{s,t})=\mathbb{E}[\mathbf{X}_{t,s}]$ thanks to the formula

$$\mathbb{E}[\mathbf{X}_{t,s}]=e^{\frac{1}{2}\int_t^s\int_t^s k(u,v)dudv}. \tag{2}$$

It appears to be actually quite a difficult question in the non-commutative setting. Eventhough it is a rather natural question to adress concerning differential equations in free probability, our first motivation came actually from standard statistical mechanics, namely from the study of the aging properties of p-spherical spin glasses. Indeed, consider a spin glass with Hamiltonian

$$H_J(x) = \sum_{p=1}^{M} \frac{a_p}{p!} \sum_{1\le i_1 \cdots i_p \le N} J_{i_1\cdots i_p} x^{i_1} \cdots x^{i_p},$$

$x = (x^i)_{1\le i\le N}$, $x^i \in \mathbb{R}$, with independent centered gaussian variables $J_{i_1\cdots i_p}$. The Langevin dynamics for this model with a smooth spherical constraint are given by the stochastic differential system

$$\mathrm{d}x_t = -f'(||x_t||^2/N)x_t \mathrm{d}t - \nabla H_J(x_t)\mathrm{d}t + \mathrm{d}B_t,$$

where B_t is a N-dimensional Brownian motion and f is a convex function. Let

$$C_N(s,t) = \frac{1}{N}\sum_{i=1}^{N} x_s^i x_t^i, \quad \mathcal{X}_N(s,t) = \frac{1}{N}\sum_{i=1}^{N} x_s^i B_t^i.$$

It was shown in [3] that the couple (C_N, χ_N) converges almost surely towards functions (C, χ). If we set

$$R(s,t) = \partial_s \mathcal{X}(s,t)$$

be the so-called response function of the system, then (see [3]), (C, G) satisfies the following integro-differential equations given, for $t \le s$, by

$$\partial_s R(s,t) = -f'(C(s,s))R(s,t) + \int_t^s R(s,u)R(u,t)\nu''(C(s,u))\mathrm{d}u,$$

$$\begin{aligned}\partial_s C(s,t) = -f'(C(s,s))C(s,t) \quad &+ \int_0^t R(t,u)\nu'(C(s,u))\mathrm{d}u \\ &+ \int_0^s R(s,u)C(t,u)\nu''(C(s,u))\mathrm{d}u,\end{aligned}$$

where the function ν is given by

$$\nu(x) = \sum_{p=1}^{M} \frac{a_p^2}{p!} x^p.$$

Here, the boundary conditions are given by $R(t,t) \equiv 1$ and $C(0,0)$ (which is known). Similar type of equations have been derived in various contexts such as the dynamics of long-range superconducting networks [6] or for other dynamical models [11].

The main question which arises in physics is to understand the long time behaviour of C, which measures the long time memory of the system and aging (see [2] for a detailed study of the easiest case $\nu(x) = \frac{c}{2}x^2$). [8] derived the same set of Schwinger-Dyson equations (for the hard sphere model where f is a function of the time variable only, chosen so that $C(t,t) \equiv 1$ for all $t \in \mathbb{R}^+$) and proposed heuristics concerning the asymptotic behavior of the solutions when t and s are large (see also [9]). However, even on a non-rigourous ground, no complete description of these asymptotics could be given, but rather scenarios about their form could be validated or dismissed. The idea is indeed to assume a form for the asymptotics of the couple (C, R) in different regimes of the time parameters (t, s) ; for instance, one can imagine that on $D_{\text{FDT}} := \{s \geq t : s - t \ll t\}$ (corresponding to the so-called FDT regime), the solutions are approximately stationnary

$$C(s,t) \equiv C_{\text{FDT}}(s-t), \quad R(s,t) \equiv R_{\text{FDT}}(s-t),$$

with a standard choice of the form $C_{\text{FDT}}(x) = Ae^{-ax}$, $R_{\text{FDT}}(x) = Be^{-ax}$, but on $D_{\text{AGING}} = \{s \geq t : t/s \in (0,1)\}$ (corresponding to an aging regime), one can expect

$$C(s,t) \equiv C_{\text{AGING}}(\frac{t}{s}), \quad R(s,t) \equiv \frac{1}{s} R_{\text{AGING}}(\frac{t}{s})$$

with a standard guess $C_{\text{AGING}}(x) = Ax^p$, $R_{\text{AGING}}(x) = Bx^q$ for some exponents p, q to be determined. Then, one checks whether this scenario is consistent with the above integro-differential system. However, the form of the intermediate phases between these different domains is hard to predict and actually determines the exponents (such as p, q) since they give the slope of the different curves at their boundary (in fact, the integrals in our system of equations keep track of all the past of the trajectories, including these intermediate phases). Hence, such a strategy can not, without an intuition about these intermediate phases, predict completely the solution.

In this paper, we shall study only the equation for the response function being given the covariance C, which asymptotics shall take the forms that we expect to encounter for the solution of the full system. Hopefully, this will unable us to perform later a bootstrap argument to solve our original problem concerning the asymptotic behaviour of the covariances of p-spherical systems for instance. At least, we hope it will shade some light on the behaviour of the response function.

If we set

$$H(s,t) = \exp(\int_t^s f'(C(u,u))\mathrm{d}u)R(s,t),$$

it is easy to check that H solves the equation (1) with $k(s,u) = \nu''(C(s,u))$. Hence, studying the asymptotics of the response function being given the covariance k is equivalent to study of the long time behaviour of the solution of (1). As a remark, we want to point out that we have no deep insight why the response function should be related with a non-commutative process ; we only realize that both evolutions are given by the same integro-differential equation. It is not clear if the full system could have such an operator interpretation.

Amazingly, our motivation brings us back to Kraichnan [13] who considered the equations (1) when trying to analyze the Schrödinger equation of a particle in a random potential; his method provides such equations for correlation functions and averaged Green's functions ! This coincidence might lies in the fact that spherical models are well suited for his expansion method, but we shall not study this question here.

Throughout this article we will assume that

Hypothesis 1.1 *We shall assume that is non negative and uniformly bounded, i.e that*

$$k(t,s) \geq 0, \quad t,s \in \mathbb{R}^+ \quad , \sup_{s,t\in\mathbb{R}^+} k(t,s) = \sup_{s\in\mathbb{R}^+} k(s,s) < \infty.$$

This hypothesis should be fulfilled by the covariances of the p-spins models.

Let us now state our main results. Our most precise estimates are obtained in the case where the covariance is stationnary, in which case the Fourier transform of the solution H to (1) is given by rather nice formulae so that we can use complex analysis to estimate H.

We have then the following dichotomy depending whether k converges to zero or not as time goes to infinity ;

Theorem 1.2 *1) Assume that the kernel k is such that there exists $a > 1$ and $C < \infty$ so that*

$$0 \leq k(u) \leq \frac{C}{(1+u)^a}.$$

Then, there exists $\lambda_c > 0$ such that

$$\exp(-\lambda_c t)H(t) \sim \frac{1}{A\Gamma(2)}, \quad t \to +\infty,$$

where

$$A = \frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda - \hat{H}k(\lambda))|_{\lambda=\lambda_c(H)} > 0.$$

2)Assume that $k(u) = c_2 + c_1 k_1(u)$*, for some positive constants* c_1*,* c_2 *and* $|k_1(u)| \leq C(1+u)^{-1}$ *for some finite* C*. Then, there exists* $\lambda_c > 0$ *such that*

$$e^{-\lambda_c t} H(t) \sim A t^{-3/2}, \quad t \to +\infty,$$

for some positive constant A*.*

This theorem is proved in Theorems 4.4 and 4.6. We can not in general compute the Lyapounov exponent λ_c except in the case where $k(u) = ce^{-\delta u}$. In this case, which we study in details in section 4.1.1, λ_c appears to be the smallest zero of a Bessel function. Interestingly, this special case has already been studied for combinatorial reasons in [4] ; the Laplace transform of H can in fact be interpreted as the generating function of random staircase polygons.

We also consider the general case and prove that

Theorem 1.3 *Assume that*

$$k(s,t) = k_1(s-t) + h(s,t)$$

with a flat function h *such that there exists a positive constant* C *and for* $T, M > 0$ *a function* $\delta(T,M)$ *such that* $\delta(T,M) \to 0$ *for all* M *when* T *goes to infinity so that*

$$\sup_{t \geq T} \sup_{|s-t| \leq M} |h(s,t) - C| \leq \delta(M,T), \qquad \sup_{s \geq t \geq T} |h(s,t)| \leq C. \tag{3}$$

Then, we claim that if we denote H_k *the solution of (1), regardless of the way* $t \geq T$ *goes to infinity*

$$\lim_{t\to\infty} \lim_{s-t\to\infty} \frac{1}{s-t} \log H_k(s,t) = \lim_{t\to\infty} \lim_{s-t\to\infty} \frac{1}{s-t} \log H_{C+k_1}(s,t) = \lambda_c(H_{C+k_1}).$$

The study of the second order correction in this general case is of course much more complicated and demand much more precise hypothesis concerning the function h. We shall come back to this issue in a forthcoming research.

The plan of the article is as follows ; we first show that $(\phi(\mathbf{X}_{s,t}), s \leq t)$ can be described as the unique solution of (1) in section 2, we then study the robustness of the first order asymptotics of H when k varies in section 3. In section 4, we consider the case where k is stationnary. The general case is considered in section 5.

2 Description of $(\phi(\mathbf{X}_{s,t}), s \leq t)$ as the unique solution of the integro-differential equation

Let L be a semicircular process with covariance k and $\mathbf{X}_{s,t}$ satisfying the linear differential equation

$$\partial_s \mathbf{X}_{s,t} = L_s \mathbf{X}_{s,t},$$

with boundary data $\mathbf{X}_{t,t} = 1$. We first remind the reader why $(\phi(\mathbf{X}_{s,t}), s \geq t)$ satisfies (1) and then that it is actually uniquely described by this property.

2.1 $(\phi(\mathbf{X}_{s,t}), s \geq t)$ satisfies (1)

Let us recall that a semi-circular variable L_t with covariance $k(t,t)$ is uniformly bounded (for the operator norm) by $2k(t,t)^{\frac{1}{2}}$. Therefore, we can use Picard formula, which can serve as a basic definition of $\mathbf{X}_{s,t}$, to write $\mathbf{X}_{s,t}$ in the form

$$\mathbf{X}_{s,t} = \sum_{n \geq 0} \int_{t \leq t_1 \cdots \leq t_n \leq s} L_{t_1} \cdots L_{t_n} dt_1 \cdots dt_n \tag{4}$$

where the serie converges uniformly with respect to the operator norm and uniformly on any times s, t in a compact interval since k is uniformly bounded.

In a combinatorial way, the fact that L is a semicircular process means that the free cumulants of L are null except the second one, which is given by k. The analogue of Wick formula for such processes is given by

$$\phi(L_{t_1} L_{t_2} ... L_{t_{2n}}) = \sum_{\sigma \in \mathrm{NCP}_n} \prod_{i \in \mathrm{cr}(\sigma)} k(t_i, t_{\sigma(i)}), \tag{5}$$

where NCP_n denotes the set of involutions of $\{1, \cdots, 2n\}$ without fixed points and without crossings and where $\mathrm{cr}(\sigma)$ is defined to be the set of indices $1 \leq i \leq 2n$ such that $i < \sigma(i)$. $\sigma \in \mathrm{NCP}_n$ when the situation $i < j < \sigma(i) < \sigma(j)$ does not occur. Therefore, $B(t_1, \cdots, t_n) := \phi(L_{t_1} \cdots L_{t_n})$ is null when n is odd and otherwise satisfies the recursion formula

$$B(t_1, \cdots, t_{2n}) = \sum_{p=1}^{2n} k(t_1, t_p) B(t_2 .... t_{p-1}) B(t_{p+1} ... t_{2n}).$$

As a consequence of (4), we get

Lemma 2.1

$$H(s,t)=\phi(\mathbf{X}_{s,t})=\sum_{n\geq 0}\int_{t\leq t_1\cdots\leq t_{2n}\leq s}\sum_{\sigma\in NCP_n}\prod_{i\in cr(\sigma)}k(t_i,t_{\sigma(i)})dt_1\cdots dt_{2n}\quad (6)$$

solves (1) and satisfies

$$H(s,t)\leq \exp(2\int_t^s k(u,u)^{1/2}du). \quad (7)$$

Proof. Indeed, by definition, $\partial_s H(s,t)$ is given by

$$\sum_{n\geq 0}\sum_{i:\sigma(i)=2n}\int_{t\leq t_1\cdots\leq t_{2n-1}\leq s}k(t_i,s)\sum_{\sigma\in \mathrm{NCP}_n\setminus\{2i-1,2n\}}\prod_{i\in \mathrm{cr}(\sigma)}k(t_i,t_{\sigma(i)})dt_1\cdots dt_{2n-1}$$

$$=\sum_{n\geq 0}\sum_{i=1}^{n}\int_{t\leq t_{2i-1}\leq s}k(t_i,s)\left(\int_{t\leq t_1\cdots\leq t_{2i-2}\leq t_{2i-1}}\sum_{\sigma\in \mathrm{NCP}_{i-1}}\prod_{j\in \mathrm{cr}(\sigma)}k(t_j,t_{\sigma(j)})dt_1\cdots dt_{2i-2}\right)$$

$$\times\left(\int_{t_{2i-1}\leq t_1\cdots\leq t_{2(n-i-1)}\leq s}\sum_{\sigma\in \mathrm{NCP}_{n-i-1}}\prod_{j\in \mathrm{cr}(\sigma)}k(t_j,t_{\sigma(j)})dt_1\cdots dt_{2(n-i)-2}\right)dt_{2i-1}$$

$$=\int_t^s k(u,s)H(u,t)H(s,u)du,$$

where in the second line we noticed that $\{i:\sigma(i)=2n\}=\{1,3,\cdots,2n-1\}$ since $\sigma\in\mathrm{NCP}_n$ and we obtained the last one by summing over the indices $1\leq i\leq n\leq\infty$. The only point which remains to prove is (7) :

$$\begin{aligned}
\phi(\mathbf{X}_{s,t}) &= \sum_{n\geq 0}\sum_{\sigma\in \mathrm{NCP}_n}\int_{t\leq t_1\cdots\leq t_{2n}\leq s}\prod_{i\in \mathrm{cr}(\sigma)}k_2(t_i,t_{\sigma(i)})\prod dt_i\\
&\leq \sum_{n\geq 0}\sum_{\sigma\in \mathrm{NCP}_n}\frac{1}{2n!}\left(\int_t^s k(u,u)^{\frac12}du\right)^{2n}\\
&= \sum_{n\geq 0}C_n\frac{1}{2n!}\left(\int_t^s k(u,u)^{\frac12}du\right)^{2n}\\
&= E(e^{\int_t^s k(u,u)^{\frac12}du S}),
\end{aligned}$$

where we used Cauchy-Schwarz inequality, where C_n denotes the Catalan number of order n (i.e the number of partitions in NCP_n), and S a standard semi-circular random variable (i.e. a random variable with law $\sigma(dx)=C\sqrt{4-x^2}dx$)

which is well known to satisfy $\mathbb{E}[S^{2n}] = C_n$. Using the fact that S is bounded by 2 uniformly, we obtain (7).

Remark 2.2: When k is uniformly bounded, say by C, we deduce

$$\phi(\mathbf{X}_{s,t}) \leq e^{2\sqrt{C}(s-t)}.$$

Moreover, a look at the previous proof shows that, if $k(t,s) \equiv C$,

$$\lim_{s-t\to\infty} \frac{1}{s-t} \log \phi(\mathbf{X}_{s,t}) = 2\sqrt{C}.$$

2.2 Uniqueness of the solution of (1)

Set

$$\mathcal{E}_M = \{f \in \mathcal{C}_b(\mathbb{R}^+ \times \mathbb{R}^+) : \quad |f(s,t)| \leq Me^{M|t-s|}\}$$

Theorem 2.3 *Assume that k is uniformly bounded on $\mathbb{R}^+ \times \mathbb{R}^+$ by some constant C. Then, for any $M \in \mathbb{R}^+$, there exists at most one solution to (1) in $\mathcal{E}_M$. Moreover if $M \geq 2\sqrt{C}$, the solution is given by (6).*

Proof. Consider two solutions in $\mathcal{E}_M$, H and $\widetilde{H}$ and set

$$\Delta(s,t) = e^{-M|s-t|}|H(s,t) - \widetilde{H}(s,t)|.$$

Then

$$\begin{aligned}
\Delta(T,t) &\leq CM \int_{t\leq u\leq s\leq T} [\Delta(s,u) + \Delta(u,t)]e^{M(s-T)} duds \\
&\leq CM \int_{t\leq u\leq s\leq T} \Delta(s,u)e^{M(s-T)} duds + C\int_{t\leq u\leq T} \Delta(u,t)du
\end{aligned}$$

Using Gronwall's lemma (Δ is bounded by hypothesis) to get rid of the last term in the above right hand side, we deduce

$$\Delta(T,t) \quad \leq \quad CMe^{C(T-t)} \int_{t\leq u\leq s\leq T} \Delta(s,u)e^{-M(T-s)} duds \tag{8}$$

Consequently, $F(T) = \sup_{0\leq t\leq T} \Delta(T,t)$ satisfies

$$F(T) \leq CMe^{(M+C)T} \int_t^T F(u)du$$

is null by Gronwall's lemma, resulting with Δ null by (8). The last part of the statement is a consequence of Remark 2.2.

Example 2.4 *Suppose that $k(t,s) = h(s)h(t)$, for some function f. In such a case, we simply take $L_t = h(t)S$ with a given semicircular variable S. Then, the solution is the 'classical' one*

$$\mathbf{X}_{s,t} = e^{S\int_t^s h(u)du},$$

and therefore,

$$\begin{aligned} H(s,t) &= \mathrm{E}(e^{S\int_t^s h(u)du}) \\ &= C\int e^{\int_t^s h(u)dux}\sqrt{4-x^2}dx. \end{aligned}$$

Consequently, when $\int_t^s h(u)du$ goes to infinity,

$$H(s,t) \approx (\int_t^s h(u)du)^{-3/2}\exp\{2\int_t^s h(u)du\}.$$

This can be compared with the classical setting where (see (2))

$$\mathbb{E}[\mathbf{X}_{s,t}] = e^{\frac{1}{2}(\int_t^s h(u)du)^2}.$$

3 Weak continuity statements

In this section, we shall investigate the robustness of the asymptotic behaviour of H when k varies. Let us first note that by (6), it is clear that

Property 3.1 *For any $t_0 \geq 0$, any covariance kernels k_1, k_2 such that $0 \leq k_1(s,t) \leq k_2(s,t)$ for all $s \geq t \geq t_0$,*

$$H_{k_1}(s,t) \leq H_{k_2}(s,t), \quad \forall s \geq t \geq t_0$$

Proposition 3.2 *Let $k_1(s,t)$ and $k_2(s,t)$ be two covariance functions such that for any $\epsilon > 0$, there is $t_\epsilon < \infty$ such that for $s \geq t \geq t_\epsilon$*

$$(1-\epsilon)k_2(s,t) \leq k_1(s,t) \leq (1+\epsilon)k_2(s,t).$$

Then, denoting by H_k the solution of (1) with kernel k, we have, uniformly for any $t > t_\epsilon$

$$\liminf_{s-t\to\infty} \frac{1}{s-t}\log H_{k_2}(s,t) + 2Ce\log(1-\epsilon) \leq \liminf_{s-t\to\infty} \frac{1}{s-t}\log H_{k_1}(s,t) \leq$$

$$\leq \limsup_{s-t\to\infty} \frac{1}{s-t}\log H_{k_1}(s,t) \leq \limsup_{s-t\to\infty} \frac{1}{s-t}\log H_{k_2}(s,t) + 2Ce\log(1+\epsilon).$$

Proof. From (6), we know that

$$H_k(s,t) = \sum_{n\geq 0} B_n^k(s,t)$$

with

$$B_n^k(s,t) = \int_{t\leq t_1\leq \cdots \leq t_{2n}\leq s} \phi(L_{t_1}\cdots L_{t_{2n}}) \prod dt_i.$$

Hence, if $A = \sum_{n\geq 1} n^{-2}$,

$$\max_{n\geq 0}(B_n^k(s,t)) \leq H_k(s,t) \leq A \max_{n\geq 0}((n+1)^2 B_n^k(s,t)). \tag{9}$$

We already noticed that

$$B_n^k(s,t) \leq \frac{(\int_t^s k(u,u)^{\frac{1}{2}}du)^{2n}}{(2n)!} C_n \leq \frac{(2C(s-t))^{2n}}{(2n)!}$$

where C is a bound on $k^{\frac{1}{2}}$ and where we used $C_n \leq 4^n$. As a consequence, using Stirling formula, for any $B > 2C$, and $s-t$ large enough,

$$\max_{n\geq B(s-t)} (n+1)^2 B_n^k(s,t) \leq (B(s-t)+1)^2 \left(\frac{eC}{B}\right)^{2B(s-t)}.$$

Therefore, fixing any $B > eC$, say $B = 2eC$, we see that there exists $M < \infty$ such that

$$\sup_{\substack{s-t>M\\ t\in\mathbb{R}^+}} \max_{n\geq 2eC(s-t)} (n+1)^2 B_n^k(s,t) \leq 1.$$

Consequently, $\sup_{\substack{s-t>M\\ t\in\mathbb{R}^+}} \max_{n\geq 0}((n+1)^2 B_n^k(s,t))$ is given by

$$\sup_{\substack{s-t>M\\ t\in\mathbb{R}^+}} \max\left\{\max_{n\leq 2eC(s-t)}((n+1)^2 B_n^k(s,t)), \max_{n\geq 2eC(s-t)}((n+1)^2 B_n^k(s,t))\right\}$$

$$\leq \max\left\{\sup_{\substack{s-t>M\\ t\in\mathbb{R}^+}} \max_{n\leq 2eC(s-t)}((n+1)^2 B_n^k(s,t)), 1\right\}.$$

But, by definition $B_0^k(s,t) \equiv 1$ so that in fact

$$\sup_{\substack{s-t>M\\ t\in\mathbb{R}^+}} \max_{n\geq 0}((n+1)^2 B_n^k(s,t)) \geq 1$$

and therefore for any $s-t > M$, any $t \in \mathbb{R}^+$,

$$\max_{n\geq 0}(n^2 B_n^k(s,t)) = \max_{n\leq 2eC(s-t)}(n^2 B_n^k(s,t)).$$

As a consequence, we deduce from (9) that for any $s-t > M$, any $t \in \mathbb{R}^+$,

$$\max_{n\geq 0}(B_n^k(s,t)) \leq H_k(s,t) \leq A(2eC(s-t))^2 \max_{n\leq 2eC(s-t)} B_n^k(s,t). \tag{10}$$

We thus deduce that, regardless of the way t goes to infinity (or not),

$$\limsup_{s-t\to\infty} \frac{1}{s-t}\log H_k(s,t) = \limsup_{s-t\to\infty}\frac{1}{s-t}\log \max_{n\leq 2eC(s-t)} B_n^k(s,t), \tag{11}$$

and

$$\liminf_{s-t\to\infty} \frac{1}{s-t}\log H_k(s,t) = \liminf_{s-t\to\infty}\frac{1}{s-t}\log \max_{n\leq 2eC(s-t)} B_n^k(s,t).$$

Now, for $t \geq t_\epsilon$, our hypothesis implies $B_n^{k_1}(s,t) \leq (1+\epsilon)^n B_n^{k_2}(s,t)$, which results with

$$\limsup_{s-t\to\infty} \frac{1}{s-t}\log H_{k_1}(s,t) \leq 2Ce\log(1+\epsilon) + \limsup_{s-t\to\infty}\frac{1}{s-t}\log H_{k_2}(s,t)$$

The same arguments apply for the lower bound.

c.q.f.d.

We now show a slightly stronger result giving the first order asymptotics of H for slowly decaying covariances

Corollary 3.3 *Suppose that k is a covariance such that for all $\epsilon > 0$, there exists $t_\epsilon < \infty$ such that for all $s \geq t \geq t_\epsilon$,*

$$Ce^{-\epsilon(s-t)} \leq k(t,s) \leq C, \tag{12}$$

for some positive constant $C > 0$. Then

$$\lim_{t\to\infty}\lim_{s-t\to\infty}\frac{1}{s-t}\log H_k(s,t) = 2\sqrt{C},$$

corresponding to the limit where $k(s,t) = C$. Moreover, if we assume additionally that $k(s,t)$ is decreasing in s and increasing in t on $s \geq t$, we also have that for any $\delta > 0$ there exists $M_\delta < \infty$ so that for (s,t) such that $(s-t)\sqrt{k(s,t)} \geq M_\delta$,

$$\frac{1}{s-t}\log H_k(s,t) \geq (2-\delta)\sqrt{k(s,t)}.$$

Thus, (12) implies

$$\lim_{s-t\to\infty}\lim_{t\to\infty}\frac{1}{s-t}\log H_k(s,t) = 2\sqrt{C}.$$

More precisely, we have in general

$$\lim_{s-t\to\infty}\sup_{k(s,t)\to 1}\frac{1}{s-t}\log H_k(s,t)=\lim_{s-t\to\infty}\inf_{k(s,t)\to 1}\frac{1}{s-t}\log H_k(s,t)=2\sqrt{C}.$$

Remark 3.4: Take $k(s,t)=C(t/s)^\alpha$ for $s\geq t$ for some $\alpha>0$. Then, it is easy to check that for any $t\geq t_\epsilon=\frac{\alpha}{\epsilon}\sup_{v>0}\frac{1}{v}\log(1+v)$, any $s\geq t$,

$$Ce^{-\epsilon(s-t)}\leq k(s,t)\leq C$$

so that the conclusions of Corollary 3.3 apply. Note that this corollary only concerns the cases where $s-t$ and t **go to infinity independently** or at most in such a way that (t/s) goes to one. In such regimes, k converges either to zero (when $s-t$ goes to infinity first) or one (when t goes to infinity first). We shall consider in section 5 the case where (12) is generalized to the case where C is not constant but a stationnary function.

Proof. By Property 3.1, we see that for $s\geq t\geq t_\epsilon$,

$$H_{Ce^{-\epsilon(s-t)}}(s,t)\leq H_k(s,t)\leq H_C(s,t).$$

Therefore,

$$\liminf_{\epsilon\to\infty}\liminf_{s-t\to\infty}\frac{1}{s-t}\log H_{Ce^{-\epsilon(s-t)}}(s,t)\leq\liminf_{t\to\infty}\liminf_{s-t\to\infty}\frac{1}{s-t}\log H_k(s,t)\leq$$

$$\leq\limsup_{t\to\infty}\limsup_{s-t\to\infty}\frac{1}{s-t}\log H_k(s,t)\leq\limsup_{t\to\infty}\limsup_{s-t\to\infty}\frac{1}{s-t}\log H_C(s,t)=2\sqrt{C}$$

where the first equality comes from the observation that $H_{Ce^{-\epsilon(s-t)}}(s,t)=H_{Ce^{-\epsilon(s-t)}}(s-t)$ so that taking t large only results in taking ϵ as small as wished and the last equality comes from Remark 2.2. We shall see in Proposition 4.2 that for any $\epsilon>0$,

$$\liminf_{s-t\to\infty}\frac{1}{s-t}\log H_{Ce^{-\epsilon(s-t)}}(s,t)=\lambda_c(\epsilon)$$

and further that $\lambda_c(\epsilon)$ converges towards $2\sqrt{C}$ as ϵ goes to zero, finishing the proof of our first result.

There is an easy argument to prove directly the second statement ; assume that $k(s,t)$ decreases in s for all t and increases in t so that $k(u,v)\geq k(t,s)$ for all $t\leq u\leq v\leq s$. Then, by (6), we find that

$$H(s,t)\geq\sum_{n\geq 0}\sum_{\sigma\in\mathrm{NCP}_n}k(s,t)^n\frac{(s-t)^{2n}}{(2n)!}=\sum_{n\geq 0}C_nk(s,t)^n\frac{(s-t)^{2n}}{(2n)!}=\ \mathbb{E}[e^{S(s-t)\sqrt{k(s,t)}}]$$

with a semicircular variable S. Thus, for any $\delta > 0$,

$$H(s,t) \geq \mathbb{P}(S > 2-\delta)e^{(2-\delta)(s-t)\sqrt{k(s,t)}}$$

yielding the estimate since $\mathbb{P}(S > 2-\delta) > 0$ for any $\delta > 0$. As a consequence, we trivially get the last point of the corollary since we already have the upper bound.

c.q.f.d

4 Asymptotic behaviour of H for stationary covariances

When $k(t,s) = k(t-s)$, (6) yields

$$H(s,t) = \sum_{n\geq 0}\sum_{\sigma\in \mathrm{NCP}_n}\int_{0\leq t_1\leq\cdots\leq t_{2n}\leq s-t} k(t_{\sigma(i)}-t_i)\prod dt_j$$

so that $H(s,t) = H(s-t)$, and (1) becomes

$$\partial_t H(t) = \int_0^t H(t-u)H(u)k(t-u)du \tag{13}$$

Consider the (eventually infinite) Laplace transform

$$\hat{H}(\lambda) = \int_0^\infty e^{-\lambda u}H(u)du.$$

Observe that since $H(u) \geq 1 > 0$ for all $u \in \mathbb{R}^+$, $\hat{H}$ is strictly decreasing. Moreover, (7) (see Remark 2.2) shows that $\hat{H}$ is finite for λ large enough. The region of convergence of the Laplace transform is of the form $(\lambda_c, +\infty)$, for some critical parameter λ_c. By assumption, the kernel k is non-negative, implying that $\hat{H}$ diverges to $+\infty$ on $(-\infty, \lambda_c)$. $\hat{H}$ is analytic on its domain of convergence and the non-negativity of k implies that the abcissa of convergence λ_c is a singularity of $\hat{H}$ (see Theorems 5a and 5b in [16]). Let

$$\lambda_c(H) = \inf\{\lambda\in\mathbb{R} : \hat{H}(\lambda) < \infty\}.$$

Note that $\lambda_c(H) < \infty$ by Remark 2.2 (in fact, $\hat{H}(+\infty) = 0$) and that $\lambda_c(H) \geq 0$ since $H \geq 1$ so that $\hat{H}(0) = +\infty$.

Note that since k is uniformly bounded,

$$\lambda_c(Hk) \leq \lambda_c(H).$$

Moreover, by (13) and using Fubini's theorem for non negative functions, we find that for any $\lambda > \lambda_c(H)$,

$$\lambda \hat{H}(\lambda) \quad = \quad 1 + \hat{H}(\lambda)(\hat{H}k)(\lambda) \tag{14}$$

We shall now show that depending whether k goes to zero or not at infinity, the asymptotic behaviour of H will be rather different. All the proofs are based on a refinment of Tauberian theorems based on analytic continuations of the function $\hat{H}$ and the following Lemma 7.2 of [2]:

Lemma 4.1 *Suppose that the Laplace transform*

$$\hat{f}(z) = \int_0^\infty e^{-zx} f(x) \mathrm{d}x,$$

of an absolutely integrable, continuous function $f(x)$, defined for $\Re(z) > 0$, has an analytic continuation on a domain S_θ of the form

$$S_\theta = \{z \in \mathbb{C}^*;\ |\arg(z)| < (\pi/2) + \theta\},$$

for some $\theta \in (0, \pi/2)$, and is such that $|\hat{f}(z)| \to 0$ as $|z| \to \infty$ in S_θ. If for some $r > 0$ and $A > 0$,

$$\limsup_{|s| \to 0,\ s \in S_\theta} |s^r \hat{f}(s) - A| = 0,$$

then

$$\limsup_{x \to +\infty} |x^{1-r} f(x) - \frac{A}{\Gamma(r)}| = 0.$$

4.1 Vanishing covariances

We start with a specific example where we can even precise the value of $\lambda_c(H)$, that is the case of exponentially vanishing covariances. We then tackle the general case.

4.1.1 Exponentially vanishing covariances

We shall study precisely the asymptotics of H in the case where $k(u) = ce^{-\delta u}$, for some $c > 0$, $\delta > 0$. We denote here in short $\lambda_c(\delta) = \lambda_c(H_{ce^{-\delta \cdot}})$. (14) gives, for $\lambda > \lambda_c(\delta)$,

$$\hat{H}(\lambda) = (\lambda - c\hat{H}(\lambda + \delta))^{-1}. \tag{15}$$

¿From this equation, we shall deduce the following

Proposition 4.2 *Assume that* $k(u) = ce^{-\delta u}$*, for some positive constants* c *and* δ*, and let* $\hat{H}(\lambda)$ *be the Laplace transform of the unique solution of (1). Then*

$$\hat{H}(\lambda) = \frac{J_{\lambda\delta^{-1}}(z)}{J_{\lambda\delta^{-1}-1}(z)}, \quad \lambda > \lambda_c(\delta), \tag{16}$$

where $J_\nu(z)$ *denotes the Bessel function of order* ν *and* $z := 2\sqrt{c}/\delta$*. Let* j_ν *denotes the smallest real positive root of* J_ν*. Then* $\lambda_c(\delta)$ *is given by the equation*

$$j_{\lambda_c(\delta)/\delta-1} = z.$$

$\lambda_c(\delta)$ *is right continuous in* δ *at zero and satisfies*

$$\lambda_c(\delta) = 2\sqrt{c} - 2.34c^{\frac{1}{3}}\delta^{\frac{2}{3}} + O(\delta).$$

Proof. First notice that c can be chosen equal to one up to replace $\hat{H}$ by $\sqrt{c}\hat{H}(\sqrt{c}.)$, δ by $\gamma = \delta/\sqrt{c}$ and therefore λ_c by $(\lambda_c/\sqrt{c})$. Moreover, it is known that if $J_\nu(z)$ is the Bessel function, and

$$h(\nu, z) = \frac{J_\nu(z)}{J_{\nu-1}(z)}$$

then

$$h(\nu, z) = \frac{\frac{1}{2}\frac{z}{\nu}}{1 - \frac{z}{2\nu}h(\nu+1, z)} \tag{17}$$

(see [17], chap. 5.6, p. 153). Moreover, $h(\nu, z)$ is uniquely determined by (17) and the boundary condition $\lim_{\nu\to\infty} h(\nu, z) = 0$. Putting $z = \frac{2}{\gamma}$ and $\nu = \frac{\lambda}{\gamma}$, we find that since $\hat{H}$ satisfies (15) with the same boundary condition than h, they are related by

$$h(\nu, 1) = \hat{H}(\gamma\nu, 1).$$

Therefore, the critical point λ_c corresponds to the largest λ such that

$$J_{\frac{\lambda}{\gamma}-1}(\frac{2}{\gamma}) = 0.$$

Since the zeros $j_{\nu,s}$ of the Bessel function increases with ν (see 9.5.2 in [1]) it follows that if $j_\nu = j_{\nu,1}$ denotes the smallest zero of the Bessel function J_ν, the equation for the critical point is

$$j_{\frac{\lambda_c(\gamma)}{\gamma}-1} = \frac{2}{\gamma}.$$

It is known (see 9.5.14 of [1]) that as ν is large,

$$j_\nu \approx \nu + 1,85575\nu^{\frac{1}{3}} + O(\nu^{-\frac{1}{3}})$$

(for the derivation of this asymptotics, see [17], chap. XV, 15.83, p. 521, Sturm's method) so that we deduce (recall that $\nu = \frac{\lambda}{\gamma}$, $\gamma = \frac{\delta}{\sqrt{c}}$),

$$\lambda_c(\delta) = 2 - 1,85575\lambda_c(\delta)^{\frac{1}{3}}\gamma^{\frac{2}{3}} + O(\gamma) = 2 - 2.34\gamma^{\frac{2}{3}} + O(\gamma).$$

Lemma 4.3 *Let $j_{\nu,n}$, $n \geq 1$, $\nu \in \mathbb{R}^+$, be the real positive zeros of J_ν arranged in increasing order. Set $\nu + 1 = \lambda\delta^{-1}$ and $z = \frac{2\sqrt{c}}{\delta}$. Then*

$$\hat{H}(\lambda) = \frac{2z}{j_\nu^2 - j_{\nu_c}^2} + 2z \sum_{n \geq 2} \frac{1}{j_{\nu,n}^2 - j_{\nu_c,1}^2}, \ \lambda > \lambda_c(\delta),$$

with

$$\hat{H}(\lambda) \approx \frac{zJ_{\nu_c+1}^2(z)}{2\nu_c \int_0^z J_{\nu_c}^2(t)\mathrm{d}t/t} \ \frac{1}{(\nu - \nu_c)}, \ \lambda \to \lambda_c(\delta),$$

with $j_\nu = j_{\nu,1}$. Moreover

$$\lim_{x \to \infty} \exp(-\lambda_c(\delta)x)H(x) = \frac{zJ_{\nu_c+1}^2(z)}{2\nu_c \int_0^z J_{\nu_c}^2(t)\mathrm{d}t/t}. \qquad (18)$$

Proof. The first identity is a classical result (see e.g. [10], vol.2, p.61). The asymptotic behavior of $\hat{H}$ when $\lambda \to \lambda_c(\delta)$ is obtained by considering the first term $(2z)/(j_\nu^2 - j_{\nu_c}^2)$ and using the analycity of the smallest positivee zero $j_{\nu,1}$ of the Bessel function when the argument is the order ν, using the asymptotics $(j_\nu^2 - j_{\nu_c}^2) \sim 2j_{\nu_c}(\partial j_\nu/\partial\nu)_{\nu=\nu_c}(\nu - \nu_c)$, and standard formulas for the derivative $\partial j_\nu/\partial\nu$ (see [17]). It remains to consider the problem of asymptotic behavior of the argument of the Laplace transform. In our situation, let

$$f(s) = \hat{H}(\lambda_c(\delta) + s) = \int_0^\infty \exp(-su)G(u)\mathrm{d}u,$$

where we set $G(u) = \exp(-\lambda_c(\delta)u)H(u)$. Then

$$f(s) = \frac{\mathrm{J}_{(\lambda_c(\delta)+s)/\delta}(z)}{\mathrm{J}_{(\lambda_c(\delta)+s)/\delta-1}(z)},$$

and the main problem is to find an analytic continuation. Coulomb [7] proved that the roots of the equation in ν, $\mathrm{J}_\nu(z) = 0$ are contained in the real axis when z is real positive, and the analytic continuation is simply given by the ratio of Bessel functions where the order ν is resticted to $\mathbb{C} \setminus \{(-\infty, \nu_c)\}$. Hence, f can be continued analytically to $\mathbb{C} \setminus \{(-\infty, \nu_c)\}$. Further, f goes to zero as $|z| \to \infty$ in this domain so that Lemma 4.1 applies, yielding (18). This follows from classical asymptotics: one uses the expansion

$$J_\nu(z) = \sum_{m \geq 0} \frac{(-1)^m (z/2)^{\nu+2m}}{m!\Gamma(\nu+m+1)},$$

which is analytic as function of $\nu \in \mathbb{C}$ (see e.g. [18]), and the Stirling's series for the Gamma function $\Gamma(w)$ when the complex argument $w \in \mathbb{C}$ is such that $|\arg(w)| \leq \pi - \triangle$, for some positive number $0 < \triangle < \pi$ to get that

$$J_\nu(z) \sim \exp(\nu + \nu \ln(z/2) - (\nu + 1/2)\ln(\nu))\sqrt{2\pi},$$

as $|\nu| \to \infty$ ([17], chap. 3).

4.1.2 Vanishing covariances ; the (almost) general case

Theorem 4.4 *Assume that the kernel k is such that there exists $a > 1$ and $C < \infty$ so that*

$$0 \leq k(u) \leq \frac{C}{(1+x)^a}.$$

Then,

$$\exp(-\lambda_c(H)t)H(t) \sim \frac{1}{A\Gamma(2)}, \quad x \to +\infty,$$

where

$$A = \frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda - \hat{H}k(\lambda))|_{\lambda=\lambda_c(H)} > 0.$$

Proof. Hereafter we denote λ_c for $\lambda_c(H)$. The proof goes as follows

1. We first show that we must have $\lambda_c = \hat{H}k(\lambda_c)$. This already entails that

$$\hat{H}(\lambda) \sim (A(\lambda - \lambda_c))^{-1}, \quad \lambda \to \lambda_c$$

and hence by Tauberian theorem (see Theorem 4.3 of chap. V in [16]),

$$\int_0^x e^{-\lambda_c(H)u} H(u)du \sim \frac{1}{A\Gamma(2)} x \quad x \to \infty.$$

2. To suppress the integral above, one has to use in general complex analysis, Tauberian theorems being then only valid under additionnal monotony properties which are not a priori satisfied here. To this end, we construct an analytic continuation h of $\hat{H}$ in a set of the form

$$\Gamma_{r,\theta} = S_\theta \backslash \mathrm{B}(\lambda_c, r) \tag{19}$$

with

$$S_\theta = \{z \in \mathbb{C} : |\arg(z - \lambda_c)| \leq \frac{\pi}{2} + \theta\} \text{ and } \mathrm{B}(\lambda_c, r) = \{z \in \mathbb{C} : |z| \leq r\}$$

for any $r > 0$ and θ small enough. We show that $h(z)$ goes to zero as $|z|$ goes to infinity and further

$$\lim_{z \in S_\theta} \sup_{|z-\lambda_c| \to 0} |zh(z) - A^{-1}| = 0.$$

We can therefore apply Lemma 4.1 and conclude.

To prove the first point, we proceed by contradiction assuming that $\lambda_c > \hat{H}k(\lambda_c)$, and constructing then an analytic continuation of $\hat{H}$ in a neighborhood of λ_c, which is a contradiction with the definition of λ_c. To do that, let us notice that under our hypothesis, $\hat{H}$ is bounded on $\lambda \geq \lambda_c$, and therefore $\hat{G}(\lambda) = \hat{H}k(\lambda)$ is bounded and continuously differentiable since we assumed $a \geq 1$. Consequently, $\hat{H}$ is also continuously differentiable. Proceeding by induction, we see that $\hat{H}$ and $\hat{G}$ are $\mathcal{C}^\infty$ at λ_c. Let $h_n = n!\hat{H}^{(n)}(\lambda_c)$ and $g_n = n!\hat{G}^{(n)}(\lambda_c)$. We now bound the h_n and g_n by using the idea of majorizing sequences following Cartan [5], chapter VII. Remark that $V(x, y) = (x - y)^{-1}$ is analytic in a neighborhood of (λ_c, g_0) such as $\{|x - \lambda_c| \leq 3^{-1}|g_0 - \lambda_c|\} \times \{|y - g_0| \leq 3^{-1}|g_0 - \lambda_c|\}$ since we assumed $g_0 \neq \lambda_c$. Therefore, there exists $r > 0$ (r can be taken equal to $3^{-1}|g_0 - \lambda_c|$ according to the choice of the above neighborhood) such that for $|x - \lambda_c| < r$ and $|y - g_0| < r$, $V(x, y) = \sum_{p,q} c_{p,q}(x - \lambda_c)^p(y - g_0)^q$ with, by Cauchy formula, a finite constant M $(= \sup_{|x-\lambda_c|<r, |y-g_0|<r} |V(x, y)|)$ such that

$$|c_{p,q}| \leq \frac{M}{r^{p+q}}.$$

Then, if we consider the formal series $h(\lambda) = \sum h_n(\lambda - \lambda_c)^n$ and $g(\lambda) = \sum_n g_n(\lambda - \lambda_c)^n$, they are formal solutions of

$$\lambda h(\lambda) = 1 + h(\lambda)g(\lambda) \qquad \Leftrightarrow \qquad h(\lambda) = V(\lambda, g(\lambda))$$

so that we find that for all $n \geq 0$,

$$h_n = P_n(g_1, \cdots, g_n; c_{p,q}) \tag{20}$$

with polynomial functions P_n with non negative coefficients. Because the polynomial functions P_n have non negative coefficients, we deduce that

$$|h_n| \leq P_n(|g_1|, \cdots, |g_n|; |c_{p,q}|).$$

Note that for all $n \geq 1$, since $a \geq 1$,

$$|g_{n+1}| = (n+1)! \int_0^\infty e^{-\lambda_c u} H(u) k(u) u^{n+1} du \leq C(n+1)! \int_0^\infty e^{-\lambda_c u} H(u) u^n du = C(n+1)|h_n|$$

so that we deduce for $n \geq 1$

$$|g_n| \leq C n P_n(|g_0|, \cdots, |g_{n-1}|; \frac{M}{r^{p+q}}) \tag{21}$$

Now, we can construct a majorizing sequence by considering the solution of

$$\partial_\lambda k(\lambda) = F(\lambda, k(\lambda))$$

with $F(x,y) = C^{-1} M (1 - \frac{x-\lambda_c}{r})^{-1} (1 - \frac{y-g_0}{r})^{-1}$ and $k(\lambda_c) = k_0 = g_0$. In fact, by the implicit function theorem the solution k exists and is unique in a neighborhood of (λ_c). Writing $k(\lambda) = g_0 + \sum_{n\geq 1} k_n (\lambda - \lambda_c)^n$, we see that $k_i \geq 0$ for all i and

$$k_{n+1} = C(n+1) P_n(k_0, \cdots, k_n; \frac{M}{r^{p+q}})$$

showing with (21) by induction that for all $n \in I\!N$,

$$|g_n| \leq k_n. \tag{22}$$

Finally, it is not hard to see that

$$k(\lambda) = g_0 + r \left(1 - \sqrt{1 + 2C^{-1} M \log(1 - \frac{\lambda - \lambda_c}{r})} \right)$$

implying that

$$k_n \leq \left(r(1 - e^{-\frac{1}{2C^{-1}M}}) \right)^{-n}$$

for some finite constant C. This conclude the proof since (21) shows that

$$|g_n| \leq \left(r(1 - e^{-\frac{1}{2C^{-1}M}}) \right)^{-n}, \tag{23}$$

so that g is an analytic continuation of $\hat{G}$ in $\{|\lambda - \lambda_c| < r(1 - e^{-\frac{1}{2C^{-1}M}})\}$ and therefore $h(\lambda) = (\lambda - g(\lambda))^{-1}$ is an analytic continuation of $\hat{H}$ in $\{|\lambda - \lambda_c| < 2^{-1} r (1 - e^{-\frac{1}{2C^{-1}M}})\}$. This contradicts the definition of λ_c. Thus $\lambda_c = \hat{G}(\lambda_c)$.

We now construct an analytic continuation of $\hat{H}$.

- Analytic continuation to $S_\theta \cap \{|\Im z| \geq R\}$ for some sufficiently large R : ¿From $\lambda_c = \hat{G}(\lambda_c)$ and the fact that $\hat{G}$ is continuously differentiable at λ_c, we see that

$$\hat{H}(\lambda) \sim (A(\lambda - \lambda_c))^{-1} \quad \lambda \to \lambda_c$$

implying by Tauberian theorem (see [16]) that

$$\int_0^x e^{-\lambda_c t} H(t)dt \sim \frac{1}{\Gamma(2)A} x, \quad x \to \infty.$$

Consequently, we see by integration by parts that if $a > 1$,

$$B := |\hat{H}k(\lambda_c)| < \infty$$

and therefore, by (14), that the analytic continuation of $\hat{H}$ to $\{\Re(z) > \lambda_c\}$ satisfies uniformly on this set

$$\hat{H}(z) = z^{-1} + o(|z|^{-1}), \quad |z| \to \infty$$

In particular, for R large enough, $z - \hat{G}(z)$ has no pole in $\Gamma_R = \{\Re(z) > \lambda_c\} \cap \mathrm{B}(\lambda_c, R)^c$. We can therefore proceed as above by majorizing sequences to see that we can extend analytically $\hat{H}$ around each point z_0 of the type $z_0 = \lambda_c + \varepsilon + iy$ with $y > R$ and $\varepsilon > 0$ and this continuation is analytic in $|z - z_0| < C|y - R|$ for some universal constant C (indeed note that here the radius r of convergence of V is of the order of the distance $|y-R|$). Further, for $y \geq 2R$ it is not hard to see that on $|z-z_0| < C|y-R|$, the continuation of $\hat{H}$ and therefore $\hat{G}$, is bounded by B. Performing such analytic continuation for every $z_0 = \lambda_c + \varepsilon + iy$ with $|y| \geq R$, we obtain an analytic continuation h, g of $(\hat{H}, \hat{G})$ on $\Gamma_R = S_\theta \cap \{|\Im(z)| \geq R\}$ with $\tan(\theta) \leq 2^{-1}C$ such that g is uniformly bounded by B. Moreover, note that since by construction, $\hat{G}$ remains uniformly bounded and the continuation satisfies $\hat{H} = (z - \hat{G})^{-1}$,

$$\hat{H}(z) = O(|z|^{-1}), \quad |z| \to \infty$$

- Analytic continuation to $\{|\arg(z - \lambda_c)| \leq \frac{\pi}{2} + \theta\} \cap \mathrm{B}(\lambda_c, r)^c \cap \mathrm{B}(\lambda_c, R)$: Again, the main issue is to control the zeros of $z - \hat{H}k(z)$. Let us study these zeroes on $\Re(z) = \lambda_c$. Observe that for such a z,

$$z - \hat{H}k(z) = 0 = \lambda_c - \hat{H}k(\lambda_c)$$

Taking the real part of both sides of this equality, we find that

$$\int_0^\infty H(u)k(u)e^{-\lambda_c u}(\cos(\Im(z)u) - 1)du = 0$$

which implies that $\Im(z) = 0$ since $Hk \geq 0$. Hence, $(\lambda_c, 0)$ is the only zero on $\Re(z) = \lambda_c$. We can again apply majorizing sequences to continue $\hat{H}$ in the neighborhood of any (λ_c, y) with $y \neq 0$ in some domain $\mathrm{B}((\lambda_c, y), r_y)$ for some $r_y > 0$. We thus obtain a continuation on $\cup_{r \leq |y| \leq R} \mathrm{B}((\lambda_c, y), r_y)$ which can be reduced to a finite union $\cup_{1 \leq i \leq L} \mathrm{B}((\lambda_c, y_i), r_{y_i})$ since we are in the compact $B(\lambda_c, R)$. Thus, since $\epsilon = \min_{1 \leq i \leq L} r_{y_i} > 0$, we obtain an analytic continuation of $\hat{H}, \hat{G}$ on

$$\{\Re(z) > \lambda_c - \epsilon\} \cap \mathrm{B}(\lambda_c, r)^c \cap \mathrm{B}(\lambda_c, R) \subset S_\theta \cap \mathrm{B}(\lambda_c, r)^c \cap \mathrm{B}(\lambda_c, R)$$

where the latter inclusion holds for $\theta \leq \theta_0 \sim \frac{\epsilon}{R}$. This finishes the construction of the continuation of $(\hat{H}, \hat{G})$.

Finally, noting that for $z \in S_\theta$ approaching λ_c, the differentiability of $\hat{G}$ shows that $z - \hat{G}(z) \sim A(z - \lambda_c)$ with $A = 1 - G'(\lambda_c) \geq 1 > 0$ since G is decreasing, which implies

$$\hat{H}(z) \sim (A(z - \lambda_c))^{-1}.$$

We can thus conclude the proof of the lemma thanks to Lemma 4.1.

4.2 Covariances with non zero limit

In this section we consider the case where

$$\lim_{t \to \infty} k(t) = C > 0.$$

We first tackle the case where the covariance decays towards this limit exponentially fast (which is somewhat simpler) and then when the speed is only algebraic.

4.2.1 Exponentially decaying covariances

Let us now assume that we have $k(u) = c_2 + c_1 e^{-\delta u}$ with $c_1, c_2 > 0$, which should correspond to the case where we consider the p-SSK model in the range t/s of order one (see the introduction or [8]). Then, we obtain

$$\lambda \hat{H}(\lambda) = 1 + \hat{H}(\lambda)(c_2 \hat{H}(\lambda) + c_1 \hat{H}(\lambda + \delta)), \quad \lambda > \lambda_c(H).$$

We can solve this equation to find that for $\lambda > \lambda_c(H)$

$$\hat{H}(\lambda) = (2c_2)^{-1}[\lambda - c_1 \hat{H}(\lambda + \delta) - \sqrt{(\lambda - c_1 \hat{H}(\lambda + \delta))^2 - 4c_2}], \qquad (24)$$

implying $\lambda_c - c_1 \hat{H}(\lambda_c + \delta) \geq 2\sqrt{c_2}$. Set $\lambda_c := \lambda_c(H)$.

We claim that λ_c is the unique positive number such that

$$\lambda_c - c_1\hat{H}(\lambda_c+\delta) = 2\sqrt{c_2}.$$

Suppose that λ_c is such that $\lambda_c - c_1\hat{H}(\lambda_c+\delta) > 2\sqrt{c_2}$. Using the analycity of $\hat{H}$ on its domain of convergence, this remains true for λ with $|\lambda-\lambda_c|<\varepsilon$, for some small enough positive constant $\varepsilon<\delta$. Let $D := \{(x,y)\in\mathbb{R}^2;\ x-c_1y > 2\sqrt{c_2}\}\subset\mathbb{R}^2$, and consider the mapping $\Psi:\ D\longrightarrow\mathbb{R}$ given by $\Psi(x,y) := (2c_2)^{-1}[x-c_1y-\sqrt{(x-c_1y)^2-4c_2}]$. Then, the function $h:\ (\lambda_c-\varepsilon,+\infty)\longrightarrow\mathbb{R}$ given by $h(\lambda) := \Psi(\lambda,\hat{H}(\lambda+\delta))$ is analytic with $h(\lambda)=\hat{H}(\lambda)$, $\forall\lambda>\lambda_c$, and provides thus an analytic continuation of $\hat{H}$ on a domain containing its domain of convergence, a contradiction with the fact that the abcissa of convergence λ_c is a singularity of $\hat{H}$ when H is non-negative (see Theorem 5b in [16]). The computation of the Laplace transform seems difficult to obtain in closed form, and the abcissa of convergence λ_c remains unknown.

Theorem 4.5 *Assume that $k(u)=c_2+c_1\exp(-\delta u)$, for positive constants c_1, c_2 and δ. Then*

$$\hat{H}(\lambda) = \frac{1}{\sqrt{c_2}} - \frac{1}{\sqrt{c_2}}(\lambda-\lambda_c)^{1/2} + o((\lambda-\lambda_c)^{1/2}),$$

and

$$\exp(-\lambda_c x)H(x)\sim Ax^{-3/2},\quad x\to+\infty,$$

for some positive constant A.

Proof. First note that

$$\lim_{\lambda\to\lambda_c}\hat{H}(\lambda) = \lim_{\lambda\to\lambda_c}h(\lambda) = (2c_2)^{-1}(\lambda_c - c_1\hat{H}(\lambda_c+\delta)) = \frac{1}{\sqrt{c_2}}.$$

Next, $\hat{H}(\lambda)-(2c_2)^{-1}(\lambda_c-c_1\hat{H}(\lambda_c+\delta))$ is given by

$$-\frac{1}{2c_2}\sqrt{(\lambda-c_1\hat{H}(\lambda+\delta))^2-4c_2},$$

with

$$\lambda - c_1\hat{H}(\lambda+\delta) - 2\sqrt{c_2} = (1-c_1\hat{H}(\lambda_c+\delta)')(\lambda-\lambda_c) + o(\lambda-\lambda_c),$$

and

$$0 < 1-c_1\hat{H}(\lambda_c+\delta)' = 1 + c_1\int_0^{+\infty} te^{-(\lambda_c+\delta)t}H(t)\mathrm{d}t < +\infty.$$

Thus,

$$\hat{H}(\lambda) \approx \frac{1}{\sqrt{c_2}} - \frac{1}{c_2^{\frac{3}{4}}}(1 - c_1\hat{H}(\lambda_c + \delta)')^{\frac{1}{2}}(\lambda - \lambda_c)^{1/2} + o((\lambda - \lambda_c)^{1/2}),$$

Moreover, when $\lambda > \lambda_c$,

$$\hat{H}'(\lambda) = \frac{(1 - c_1\hat{H}'(\lambda + \delta))}{2c_2}(1 - \frac{(\lambda - c_1\hat{H}(\lambda + \delta))}{\sqrt{(\lambda - c_1\hat{H}(\lambda + \delta)^2 - 4c_2}}),$$

with

$$\sim -\frac{(1 - c_1\hat{H}'(\lambda_c + \delta))^{1/2}}{(2c_2)^{3/4}}(\lambda - \lambda_c)^{-1/2}, \quad \lambda \to \lambda_c. \qquad (25)$$

To prove the theorem, we need as for the proof of Theorem 4.4, to continue $\hat{H}$ (and therefore $\hat{H}'$) analytically on sets of the form (19). We in fact continue it on $[S_\theta \cap B(\lambda_c, R) \backslash B(\lambda_c, r)] \cup \{|\Im(z)| \geq R\}$ for some sufficiently large R and small θ. Let

$$\triangle_z = (z - \delta - c_1\hat{H}(z))^2 - 4c_2$$

with $\triangle_{\lambda_c+\delta} = 0$ (see above). First notice that $|\hat{H}(z)| \leq \int_0^\infty \exp(-\Re(z)x)H(x)\mathrm{d}x :=$ A, so that $\hat{H}$ is uniformly bounded on $\{\Re(z) \geq \lambda_c\}$. Therefore, for R large and any z such that $\Re(z) > \lambda_c - \delta$ and $z \in \mathrm{B}(\lambda_c, R)^c = \{z : |z - \lambda_c| \geq R\}$, $\Delta_{z+\delta} \approx z^2 + O(1) \neq 0$. Hence, if we set, for $\{\Re(z) \geq \lambda_c - \frac{1}{2}\delta\} \cap \{|\Im(z)| \geq R\}$

$$h_1(z) = (2c_2)^{-1}[z - c_1 H(z + \delta) - \sqrt{\Delta_{z+\delta}}$$

is analytic and thus provides an analytic continuation of $\hat{H}$. Further, note that if R is large enough,

$$\sup_{z \in B(\lambda_c, R)^c \cap \{\Re(z) \geq \lambda_c - \frac{1}{2}\delta\}} |h_1(z)| \leq \sup_{z \in B(\lambda_c, R)^c \cap \{\Re(z) \geq \lambda_c\}} |\hat{H}(z)| := A.$$

Indeed,

$$\begin{aligned} |h_1(z)| &= |z - c_1\hat{H}(z + \delta)|^{-1}|1 + (1 - 4c_2(z - c_1\hat{H}(z + \delta))^{-1})^{\frac{1}{2}}|^{-1} \\ &\leq (|z| - c_1 A)^{-1}|1 + (1 - 4c_2(|z| - c_1 A)^{-1})^{\frac{1}{2}}|^{-1} \leq A, \end{aligned}$$

where the last inequality holds for R large enough. ¿From this formula, we may proceed by induction to construct an analytic continuation of $\hat{H}$ on $|\Im(z)| \geq R$ by arguing by induction that $\Delta_{z+\delta}$ does not vanish. This continuation remains uniformly bounded by A.

We next show that $\Delta_{z+\delta} \neq 0$ for $z \in \mathrm{B}(\lambda_c, R) \cap S_\theta$ for θ small enough. Indeed, the analycity of $\hat{H}$ on its domain implies then that the compact intersection $D_{u,R}$ of $\mathrm{B}(\lambda_c, R)$ with $\{z \in \mathbb{C};\ \Re(z) \geq \lambda_c + u\}$, $u > 0$, contains only a finite number of roots of the equation $\triangle_z = 0$, and therefore $\Delta_{z+\delta}$ has only finitely many roots in $\mathrm{B}(\lambda_c, R) \cap S_\theta$. We can thus choose $\theta_0 > 0$ such that $\Delta_{z+\delta} \neq 0$ for $z \in \mathrm{B}(\lambda_c, R) \cap S_\theta$ when $\theta \leq \theta_0$. As a consequence, if we let for $\varepsilon > 0$ small enough, the domain Γ^1_ε be given by

$$\Gamma^1_\varepsilon = \{z \in \mathbb{C} \setminus \{\lambda_c\};\ \lambda_c - \delta + \varepsilon < \Re(z) < \lambda_c + \varepsilon\} \cap S_{\theta_0},$$

and define the function $\Psi(z)$ on this domain as

$$\Psi(z) = (2c_2)^{-1}[z - c_1\hat{H}(z+\delta) - \sqrt{\triangle_{z+\delta}}].$$

Then Ψ is analytic and, from (24), coincides with $\hat{H}$ on the band $\{z \in \mathbb{C};\ \lambda_c < \Re(z) < \lambda_c + \varepsilon\}$, and provides thus an analytic continuation of $\hat{H}$ on $\Xi_\varepsilon := \Gamma^1_\varepsilon \cup \{\Re(z) \geq \lambda_c\}$.

At the end of the day, we have constructed an analytic continuation of $\hat{H}$ on S_{θ_0}. Further, because it remains uniformly bounded, we also see that for large $|z|$,

$$|\hat{H}(z)| \approx O(\frac{1}{|z|}).$$

Consequently, $\hat{H}'$ can also be extended analytically to S_{θ_0} and its continuation remains uniformly bounded too. As a consequence,

$$|\hat{H}(z)| \approx O(\frac{1}{|z|})$$

and $|\hat{H}'(z)| \approx O(\frac{1}{|z|^2})$ by (25) We can therefore apply Lemma 4.1 and conclude.

c.q.f.d.

4.2.2 Algebraically decaying covariances

We consider here the case where the stationary covariance takes the form $k(u) = c_2 + c_1 k_1(u)$, for some positive constants c_1, c_2 and $|k_1(u)| \leq (1+u)^{-1})$ for some $a \geq 1$. Set for convenience $\lambda_c = \lambda_c(H)$, with $\lambda_c = \lambda_c(Hk)$. The basic relation becomes, for $\lambda > \lambda_c$,

$$\lambda\hat{H}(\lambda) = 1 + c_2\hat{H}(\lambda)^2 + c_1\hat{G}(\lambda)\hat{H}(\lambda), \tag{26}$$

where we set

$$\hat{G}(\lambda) = \int_0^\infty \exp(-\lambda u) H(u) k_1(u) \mathrm{d}u,$$

which converges for $\lambda > \lambda_c$, the Laplace transform of the function $G(u) = H(u)k_1(u)$.

We shall prove that

Theorem 4.6 *1. λ_c is solution of the equation*

$$\lambda_c - c_1 \hat{G}(\lambda_c) = 2\sqrt{c_2}.$$

2.

$$\hat{H}(\lambda) = \frac{1}{\sqrt{c_2}} - \frac{1}{\sqrt{c_2}}(\lambda - \lambda_c)^{1/2} + o((\lambda - \lambda_c)^{1/2}),$$

and

$$e^{-\lambda_c t} H(t) \sim A t^{-3/2}, \quad t \to +\infty,$$

for some positive constant A.

Proof. Note that $\hat{H}$ is uniformly bounded on $\lambda \geq \lambda_c$ by $\sqrt{c_2}^{-1}$ so that the integral defining $\hat{G}$ is absolutely convergent and $\lim_{\lambda \to \lambda_c} \hat{G}(\lambda)$ exists. (26) also gives the equation in $\hat{H}$, $c_2 \hat{H}^2 + (c_1 \hat{G} - \lambda)\hat{H} + 1 = 0$, $\lambda > \lambda_c$, showing that the discriminant $\triangle_\lambda = (\lambda - c_1 \hat{G})^2 - 4c_2$ is non-negative. Thus

$$\lambda_c - c_1 \hat{G}(\lambda_c) \geq 2\sqrt{c_2} \tag{27}$$

and for $\lambda > \lambda_c$,

$$\hat{H}(\lambda) = \frac{1}{2c_2}(\lambda - c_1 \hat{G}(\lambda) - \sqrt{\triangle_l}), \quad \lambda > \lambda_c, \tag{28}$$

where the branch was chosen to satisfy the condition $\lim_{\lambda \to +\infty} \hat{H}(\lambda) = 0$ and $\triangle_l = (\lambda - c_1 \hat{G}(\lambda))^2 - 4c_2$.

We can proceed exactly as in the proof of theorem 4.4 ; we prove that (27) is an equality by contradiction using majorizing sequences. The analytic continuation is also obtained similarly.

5 More general limiting behaviour

Let us consider the case where k has a sufficiently flat part around the diagonal and a stationnary part. Consider a two times non negative function h such that, there exists a positive constant C and for $T, M > 0$ a function $\delta(T, M)$ such that $\delta(T, M) \to 0$ for all M when T goes to infinity so that

$$\sup_{t \geq T} \sup_{|s-t| \leq M} |h(s,t) - C| \leq \delta(M, T), \qquad \sup_{s \geq t \geq T} |h(s,t)| \leq C. \tag{29}$$

Note that the second condition is a consequence of the first when h is a covariance, which we shall not need to assume. For instance, it is clear that such an assumption is verified by the ratio

$$h(s,t) = C \left(\frac{t}{s} \right)^a \qquad \text{for } t < s,$$

with some $a \geq 0$ or any linear combination of such functions.

Then, we claim that

Theorem 5.1 *Let k be a covariance kernel such that*

$$k(s,t) = k_1(s-t) + h(s,t)$$

with h satisfying (29) and k_1 is a non negative function. Then, regardless of the way t goes to infinity

$$\lim_{t \to \infty} \lim_{s-t \to \infty} \frac{1}{s-t} \log H_k(s,t) = \lim_{t \to \infty} \lim_{s-t \to \infty} \frac{1}{s-t} \log H_{C+k_1}(s,t) = \lambda_c(H_{C+k_1}).$$

Hence, this theorem shows that the first order asymptotics of H are only governed by its stationnary part. As a direct consequence,

Corollary 5.2 *Let h satisfying (29). Regardless of the way $t \geq T$ goes to infinity, if $h \geq 0$,*

$$\lim_{s-t \to \infty} \frac{1}{s-t} \log H_h(s,t) = 2\sqrt{C}$$

and

$$\lim_{s-t \to \infty} \frac{1}{s-t} \log H_{h+C'e^{-\delta|s-t|}}(s,t) = \lambda_c(H_{C+C'e^{-\delta \cdot}}).$$

Proof of theorem 5.1 : By property 3.1 and the second hypothesis in (29)

$$H_k(s,t) \leq H_{C+k_1}(s,t)$$

resulting with the announced upper bound. For the lower bound, note that since k is non negative, for any $n \in \mathbb{N}$, any $K \in \mathbb{N}$,

$$\begin{aligned} H(s,t) &\geq \sum_{\sigma \in \mathrm{NCP}_n} \int_{t \leq t_1 \cdots t_{2n} \leq s} \prod_{i=1}^{n} k(t_i, t_{\sigma(i)}) \prod_{j=1}^{2n} dt_j \\ &\geq \sum_{\sigma \in \mathrm{NCP}_n^K} \int_{t_1, \cdots, t_{2n} \in \Delta_n^K} \prod_{i=1}^{n} k(t_i, t_{\sigma(i)}) \prod_{j=1}^{2n} dt_j. \end{aligned} \tag{30}$$

Here, NCP_n^K are the elements of NCP_n where partitions occur only inside the boxes $[2Kp, 2K(p+1)]$ for $p \in \{0, \cdots, [\frac{n}{K}] - 1\}$ or $[2K[\frac{n}{K}], 2n]$. In other words, crossing between these boxes are prohibited and NCP_n^K is given by the set of non-crossing involutions of NCP_n such that $\sigma|_{[2Kp, 2K(p+1)]} \in \mathrm{NCP}_K$, $\forall 0 \leq p \leq [\frac{n}{K}] - 1$, and $\sigma|_{[2K[\frac{n}{K}], 2n]} \in \mathrm{NCP}_{n-K[\frac{n}{K}]}$. Moreover,

$$\Delta_n^K = \big\{ t + (s-t)\frac{pK}{n} \leq t_{2Kp+1} \leq \cdots \leq t_{2K(p+1)} \leq t + (s-t)\frac{(p+1)K}{n}, 0 \leq p \leq [\frac{n}{K}] - 1$$

$$t + (s-t)\frac{K}{n}[\frac{n}{K}] \leq t_{2(n-K[\frac{n}{K}])+1} \cdots \leq t_{2n} \leq s \big\}.$$

Observe that by construction, when $\sigma \in \mathrm{NCP}_n^K$, for all i, t_i and $t_{\sigma(i)}$ belong to the same box of the partition Δ_n^K. Hence by our hypothesis, for $t \geq T$, for all $\sigma \in \mathrm{NCP}_n^K$, all $\mathbf{t} \in \Delta_n^K$, all $i \in \{1, \cdots, n\}$,

$$k(t_i, t_{\sigma_i}) \geq k_1(t_{\sigma_i} - t_i) + \inf_{|t'-s'| \leq \frac{K}{n}(s-t)} h(t', s') \geq k_1(t_{\sigma_i} - t_i) + C - \delta$$

provided $\delta(t, \frac{K}{n}(s-t)) \leq \delta$. Therefore, we deduce from (30) that $H(s,t)$ is larger than

$$\prod_{p=1}^{[\frac{n}{K}]} \left(\sum_{\sigma \in \mathrm{NCP}_K} \int_{t+(s-t)\frac{pK}{n} \leq t_{2Kp+1} \leq \cdots \leq t_{2K(p+1)} \leq t+(s-t)\frac{(p+1)K}{n}} \prod_{i=1}^{K} (k_1(t_{\sigma_i} - t_i) + C - \delta) \prod_{i=1}^{2K} dt_i \right)$$

$$\times (C-\delta)^{n-K[\frac{n}{K}]} C_{n-K[\frac{n}{K}]} \frac{[(s-t)(1 - \frac{K}{n}[\frac{n}{K}])]^{2(n-K[\frac{n}{K}])}}{2(n-K[\frac{n}{K}])!},$$

where in the last line we bounded below the term corresponding to the indices betwen $2[\frac{n}{K}]K$ and n with the convention $0^0 = 1$. It is not hard to see that we can neglect this correction term (indeed, we shall take later n of order $s - t$ and K large, but independent of $s - t$). As a consequence of the above lower bound, we have that if we define $B^K_{C-\delta+k_1}(\frac{(s-t)K}{n})$, $\delta > 0$, by

$$\sum_{\sigma\in \mathrm{NCP}_K} \int_{t+(s-t)\frac{pK}{n}\le t_{2Kp+1}\le\cdots\le t_{2K(p+1)}\le t+(s-t)\frac{(p+1)K}{n}} \prod_{i=1}^{K}(k_1(t_{\sigma_i}-t_i)+C-\delta)\prod_{i=1}^{2K} dt_i,$$

then, for any $K, n, s-t, t$ such that $\delta(t, \frac{(s-t)}{n}K) \le \delta$

$$\begin{aligned} H(s,t) &\ge [B^K_{C-\delta+k_1}(\frac{(s-t)K}{n})]^{\frac{n}{K}} \\ &= [B^K_{C-\delta+k_1}(u)]^{\frac{s-t}{u}} \end{aligned}$$

where we have set $u = \frac{s-t}{n}K$. Now, using Jensen's inequality when $\frac{n}{K} = \frac{s-t}{u} > 1$, we deduce for any $C' > 0$, $n > K$, t, s so that $\delta(t, \frac{(s-t)}{n}K) \le \delta$,

$$\begin{aligned} H(s,t) &\ge \frac{1}{2C'eu}\sum_{K\le 2C'eu}[B^K_{C-\delta+k_1}(u)]^{\frac{s-t}{u}} \\ &\ge [\frac{1}{2C'eu}\sum_{K\le 2C'eu}B^K_{C-\delta+k_1}(u)]^{\frac{s-t}{u}} \end{aligned} \tag{31}$$

Recall that if $C' = C + ||k||_\infty$ we already observed that with $A = \sum n^{-2}$

$$H_{C-\delta+k_1}(u) \le A \max_{n\le 2C'eu} n^2 B^n_{C-\delta+k_1}(u) \le A(2C'eu)^2 \sum_{n\le 2C'eu} B^n_{C-\delta+k_1}(u). \tag{32}$$

Thus, we deduce from (31) that

$$H(s,t) \ge [\frac{1}{A(2C'eu)^3}H_{C-\delta+k_1}(u)]^{\frac{s-t}{u}} \tag{33}$$

where we have used (32) in the last line. Now, for any $\delta > 0$ by section 4, there exists $\lambda_c(C-\delta+k_1) > 0$ such that

$$\lim_{u\to\infty}\frac{1}{u}\log H_{C-\delta+k_1}(u) = \lambda_c(C-\delta+k_1)$$

so that we arrive at, for any $\epsilon > 0$, for $u \ge u(\epsilon)$ large enough, $\frac{s-t}{u} > 1$, $\delta(u,t) < \delta$

$$H(s,t) \ge [\frac{1}{A(2Ceu)^3}]^{\frac{s-t}{u}} e^{(\lambda_c(C-\delta+k_1)-\epsilon)(s-t)} \tag{34}$$

which shows, by taking first $s-t$ going to infinity and then t going to infinity, that

$$\liminf_{s-t\to\infty} \frac{1}{s-t} \log H(s,t) \geq \lim_{\delta\downarrow 0} \lambda_c(C-\delta+k_1).$$

Property (3.2) completes the proof since $k_1 \geq 0$ implies $C-\delta+k_1 \geq (1-\delta C^{-1})(C+k_1)$.

c.q.f.d

Acknowledgments We are very grateful to P. Gerard and R. Speicher for cheerful and motivating discussions.

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