Even Eight on a Kummer Surface

by

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A maman, papa et Haleh
I came to Michigan with little knowledge but with the strong conviction that learning algebraic geometry would lead me to the most beautiful concepts of mathematics. I am indebted to my advisor, Igor Dolgachev, for teaching me the subject and showing me that it is more beautiful than I ever thought. I also would like to thank him for his help during these past five years.

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CHAPTER I

Introduction

The notion of even set arises naturally in classical algebraic geometry to address the problem of finding the maximal number of nodes on a surface of degree $d$ in $\mathbb{P}^3$ ([Bea04]). In degree four, the answer to this question is sixteen and such a surface is called a Kummer surface ([Nik75]).

More precisely, an even set on a smooth surface $Y$ is a set of disjoint nodal curves, $C_1, \ldots, C_k$, whose sum is divisible by two in the Picard group of $Y$. This concept illustrates the possible relations among the exceptional curves resolving nodal singularities. Associated to an even set, there is a standard construction that consists of taking the double cover branched along the even set and blowing down its preimage. It gives rise to the diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{\epsilon} & X \\
\downarrow{\psi} & & \downarrow{\tau} \\
Y & \xrightarrow{} & X \\
\end{array}
$$

In particular, when $Y$ is the minimal resolution of a surface of degree four in $\mathbb{P}^3$ with $k$ nodes, it is a $K3$ surface with $k$ disjoint nodal curves. It is then natural to ask when such a set is even ([Nik75], [Bar02]) as well as for a classification of the surfaces $X$ arising from this construction.

As a first step towards this classification, Nikulin showed that even sets on $K3$ surfaces consist of either eight or sixteen curves and that the surface $X$ obtained as above is either a $K3$ surface or an abelian surface [Nik75].

The case where the even set consists of sixteen curves is of special interest as it recovers a well known construction in the theory of $K3$ surfaces. Indeed the diagram above realizes $Y$ as the minimal resolution of the quotient of an abelian surface $A$ by its involution automorphism, $\iota(a) = -a$. In other words, $Y$ is the Kummer surface $\text{Kum}(A)$ associated to $A$. Hosono, Liang, Oguiso and Yau showed in [HLOY03], that the set of all abelian surfaces associated to $\text{Kum}(A)$ is finite up to isomorphism and that it can be identified with the set of Fourier-Mukai partners of $A$.

The purpose of this dissertation is to study the other case: the even sets consisting of eight nodal curves on a Kummer surface. Equivalently, it aims to describe all $K3$
surfaces admitting a symplectic involution such that the quotient is birational to a fixed Kummer surface.

This classification builds on the work of Shioda and Inose ([SI77]) who consider K3 surfaces of Picard number 20. In this case, they constructed an involution on such a K3 surface whose quotient is birational to a Kummer surface. Moreover, they showed that a natural map between the transcendental lattices of the K3 surface $X$ and the associated abelian surface $A$ of the Kummer surface is a Hodge-isometry. A K3 surface with an involution inducing such a Hodge isometry is said to have a Shioda-Inose structure. Later, Morrisson gave a criterion for an arbitrary K3 surface to have a Shioda-Inose structure.

The first goal of this dissertation is to extend the classification of K3 surfaces admitting an involution whose quotient is birational to a Kummer surface to the general case, that is, to K3 surfaces that may not have a Shioda-Inose structure. We show that in the general case, there exists a rational map of degree two $X \rightarrow \operatorname{Kum}(A)$ if and only if there exists an embedding of transcendental lattices $T_X \hookrightarrow T_A$ with $T_A/T_X \simeq (\mathbb{Z}/2\mathbb{Z})^\alpha$, $\alpha \leq 4$ which preserves the Hodge decompositions of the lattices (Theorem III.9). The case where $\alpha = 0$ recovers the case of having a Shioda-Inose structure.

In the second part of this dissertation, we give a geometrical description of families of K3 surfaces $X$ satisfying our criterion. Using the work of Naruki ([Nar91]) who exhibited an actual even eight on a general Kummer surface whose associated K3 surface has a Shioda-Inose structure, we construct new even eights whose associated K3 surfaces do not admit a Shioda-Inose structure but yet are geometrically intimately related to a K3 surface with a Shioda-Inose structure (Lemmas IV.5, IV.12). Indeed this construction provides new examples of torsors over their Jacobian fibrations (Theorem IV.18). There is strong evidence that this relation could be generalized to larger families of K3 surfaces.

For certain K3 surfaces, the symplectic involution can be explained in terms of its action on a specific elliptic fibration. We find on the K3 surface $X$ an elliptic fibration with a section such that the involution is obtained as the composition of the natural involution on the general fiber, mapping $x$ to $-x$ with an involution acting trivially on the Picard group of $X$ (Theorems IV.8, IV.13).

Interestingly in each of these cases, the induced rational map $X \rightarrow Y$ decomposes as

$$
\begin{array}{ccc}
X & \xrightarrow{\zeta} & \Omega \\
\downarrow{\tau} & & \downarrow{\varphi} \\
Y & \xrightarrow{\rho} & \mathbb{P}^2
\end{array}
$$

where $\rho$ is the double cover of $\mathbb{P}^2$ branched along a reducible sextic $\Sigma$ decomposable as a (possibly degenerate) quartic $\Gamma$ and a conic $\Delta$, and where $\varphi$ and $\zeta$ are the double covers branched along $\Gamma$ and $\varphi^*(\Delta)$ respectively (Theorem IV.9, Lemma IV.15).

The last part of this dissertation deals with K3 surfaces covering a Kummer surface.
that are themselves Kummer surfaces. More precisely, we study even eights that are part of a set of sixteen disjoint nodal curves. We show that the corresponding double covers give rise to a commutative diagram

\[
\begin{array}{c}
B \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
X = \text{Kum}(B) \xrightarrow{\tau} \text{Kum}(A) = Y
\end{array}
\]

where \( q \) is an isogeny of degree two. We illustrate this phenomenon when \( Y \) is a general Kummer surface and are able to show that the map \( X \xrightarrow{\tau} Y \) corresponds again to a decomposition of the standard sextic associated to a Kummer surface into a quartic and a conic (Theorem V.8).
CHAPTER II

Lattices and $K3$ Surfaces

2.1 Elements of Lattice Theory

**Definition II.1.** A lattice $L$ is a free $\mathbb{Z}$-module of finite rank together with an integral bilinear symmetric form, $b : L \times L \to \mathbb{Z}$. The lattice $L$ is non-degenerate if $b$ is non-degenerate.

Let $L$ and $M$ be lattices, an isomorphism of $\mathbb{Z}$-modules between $L$ and $M$ preserving the bilinear forms is called an isometry. The group of self-isometries of $L$ is denoted by $O(L)$.

An embedding of lattices $M \hookrightarrow L$ is an homomorphism of $\mathbb{Z}$-modules preserving the bilinear forms. We say that the embedding is primitive if the quotient $L/M$ is torsion free.

Let $L$ be an non-degenerate lattice over $\mathbb{Z}$ and let $L^* = \text{Hom}(L, \mathbb{Z})$ be its dual, there is a natural embedding of $\mathbb{Z}$-modules

$$L \hookrightarrow L^*, \quad x \mapsto b(x, \cdot).$$

A non-degenerate lattice $L$ is unimodular if $L^*/L \cong \{0\}$. If $L$ is non-degenerate, then its signature, $(l_+, l_-)$, is the signature of the bilinear form $b$ over the real vector space $L \otimes \mathbb{R}$. A non-degenerate lattice $L$ is positive definite if $l_- = 0$, negative definite if $l_+ = 0$, and indefinite otherwise.

A lattice $L$ is even if the quadratic form $q(x) = b(x, x)$ takes only even values. It is odd if $q(x)$ takes also some odd values.

If $L$ is an non-degenerate even lattice, then the $\mathbb{Q}$-valued quadratic form on $L^*$ induces a quadratic form on the finite abelian group $D_L = L^*/L$, $q_L : D_L \to \mathbb{Q}/2\mathbb{Z}$, which is well defined mod $2\mathbb{Z}$ and a bilinear form $b_L : D_L \times D_L \to \mathbb{Q}/\mathbb{Z}$. 
<table>
<thead>
<tr>
<th>Lattice</th>
<th>Rank</th>
<th>Dynkin diagram</th>
<th>Discriminant Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>$n$</td>
<td>$\bullet \cdots \bullet$</td>
<td>$\mathbb{Z}/(n+1)\mathbb{Z}$</td>
</tr>
<tr>
<td>$D_{2n}$</td>
<td>$2n$</td>
<td>$\bullet \cdots \bullet$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$</td>
</tr>
<tr>
<td>$D_{2n+1}$</td>
<td>$2n+1$</td>
<td>$\bullet \cdots \bullet$</td>
<td>$\mathbb{Z}/4\mathbb{Z}$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$6$</td>
<td>$\bullet \bullet$</td>
<td>$\mathbb{Z}/3$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$7$</td>
<td>$\bullet \bullet \bullet$</td>
<td>$\mathbb{Z}/2$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$8$</td>
<td>$\bullet \bullet \bullet \bullet$</td>
<td>${0}$</td>
</tr>
</tbody>
</table>

Table 2.1:

The pair $(D_L, q_L)$ is called the discriminant form of $L$, $\text{discr}(L)$ denotes the order of $D_L$ and $l(D_L)$ is its minimum number of generators. If $L$ and $M$ are even non-degenerate lattices, then $D_{L\oplus M} \simeq D_L \oplus D_M$.

**Examples II.2.**

1. The hyperbolic plane $U$ is a lattice of rank two whose bilinear form is given by the matrix

   \[
   \begin{pmatrix}
   0 & 1 \\
   1 & 0 
   \end{pmatrix}
   \]

   $U$ is an even unimodular lattice of signature $(1, 1)$.

2. $A_n$, $D_n$, $E_6$, $E_7$ and $E_8$ are the positive definite lattices generated by the vertices of the Dynkin diagrams in Table 2.1. The bilinear form is defined on the vertices by the rule

   \[
   b(e_i, e_j) = \begin{cases} 
   2 & \text{if } e_i = e_j \\
   -1 & \text{if } e_i \neq e_j \text{ and } e_i \text{ and } e_j \text{ are joined by an edge} \\
   0 & \text{otherwise}
   \end{cases}
   \]

3. If $L$ is a lattice, then $L(m)$ for $m \in \mathbb{Z}$ denotes the same free $\mathbb{Z}$-module as $L$, with the bilinear form $b_{mL}(x, y) = mb_L(x, y)$.

4. $\langle k \rangle$ denotes the lattice of rank one such that $b(x, x) = k$ for any generator $x$ of $\langle k \rangle$. 

Remark II.3. The triplet \((l_+, l_-, q_L)\) is invariant under isometry. In general, it does not determine the isomorphism class of a lattice but it is known ([CS99]) that if \(L\) is indefinite of rank \(r\) and if \(|D_L| \leq d_0(r)\) for \(d_0\) given by the table below, then it does.

<table>
<thead>
<tr>
<th>(r)</th>
<th>2</th>
<th>3</th>
<th>4, 6, 8, ...</th>
<th>5, 7, 9, ...</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d_0)</td>
<td>17</td>
<td>128</td>
<td>(5^{(1)})</td>
<td>(2 \cdot 5^{(1)})</td>
</tr>
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**Overlattices**

An embedding \(S \hookrightarrow S'\) of even lattices with finite cokernel is an *overlattice*. The overlattices \(S \hookrightarrow S'\) and \(S \hookrightarrow S''\) are isomorphic if there exists an isometry of \(S\) extending to an isomorphism of lattices of \(S'\) and \(S''\).

**Proposition II.4.** Let \(S \hookrightarrow S'\) be an overlattice of even non-degenerate lattices, then

\[ |D_S| = |S' : S|^2 |D_{S'}|. \]

**Proof.** For a choice of bases, the inclusion \(S \hookrightarrow S'\) is given by an integral matrix \(G\) satisfying

\[ G^t M_{S'} G = M_S \quad \text{and} \quad |\det(G)| = |S'/S| \]

and where \(M_S\) and \(M_{S'}\) are the symmetric integral matrices of the bilinear forms in these bases. \(\square\)

Consider the chain of embeddings \(S \hookrightarrow S' \hookrightarrow S^{*'} \hookrightarrow S^*\). Let \(H_{S'} = S'/S\), then

\[ H_{S'} \subset S^{*'}/S \subset D_S \quad \text{and} \quad (S^{*'}/S)/H_{S'} \simeq D_{S'}. \]

The following proposition is a central result in the theory of lattices and we will use it on numerous occasions.

**Proposition II.5.** [Nik79] 1) The correspondence \(S' \rightarrow H_{S'}\) determines a bijection between even overlattices of \(S\) and isotropic subgroups of \(D_S\).

2) \(H_{S'}^{\perp} \simeq S^{*'}/S\) and \(q_{S}(H_{S'}^{\perp}/H_{S'}) \simeq q_{S'}\).

3) Two even overlattices \(S \hookrightarrow S'\), \(S \hookrightarrow S''\) are isomorphic if and only if the isotropic subgroups of \(D_S\), \(H_{S'}\) and \(H_{S''}\) are conjugate under some isometry of \(S\).

**Unimodular Lattices**

**Theorem II.6.** [Mil58] Let \(L\) be an indefinite unimodular lattice. If \(L\) is odd, then

\[ L \simeq \langle 1 \rangle^m \oplus \langle -1 \rangle^n \]

for some \(m\) and \(n\). If \(L\) is even, then

\[ L \simeq U^m \oplus E_8(\pm 1)^n \]

for some \(m\) and \(n\). In particular, the signature and parity of \(L\) determine \(L\) up to isometry.
Lemma II.7. [Nik79] Let $S \hookrightarrow L$ be a primitive embedding of non-degenerate even lattices, and suppose that $L$ is unimodular. Then there exists an isomorphism of groups preserving the bilinear form

$$(D_{S^\perp}, q_{S^\perp}) \simeq (D_S, -q_S).$$

In order to state necessary and sufficient conditions for the existence of a primitive embedding of an even lattice $S$ into an unimodular lattice, we need to introduce one more notion.

A $p$-adic lattice and its discriminant form are defined in the same way as for an integral lattice, only replacing $\mathbb{Z}$ by $\mathbb{Z}_p$. Let $S$ be a lattice over $\mathbb{Z}$, for every prime number $p$, $q_p$ denotes the restriction of the discriminant form $q_S$ to $(D_S)_p$, the $p$-component of $D_S$. There exists an unique $p$-adic lattice $K(q_p)$ whose discriminant form is isomorphic to $((D_S)_p, q_p)$ and whose rank is equal to $l((D_S)_p)([Nik79])$. The $p$-adic number $\text{discr} K(q_p)$ is then the determinant of a symmetric matrix associated to the lattice $K(q_p)$. It is therefore well defined mod $(\mathbb{Z}_p)^2$.

Theorem II.8. [Nik79] There exists a primitive embedding of the even lattice $S$, with invariants $(t_+, t_-, q)$ into the unimodular lattice $L$ with signature $(l_+, l_-)$ if and only if the following conditions are simultaneously satisfied:

1. $l_+ - l_- \equiv 0 \pmod{8}$,

2. $l_- - t_- \geq 0$, $l_+ - t_+ \geq 0$, $l_+ + l_ - - t_+ - t_- \geq l(D_S)$,

3. $(-1)^{l_+ - t_+}|D_S| \equiv \pm \text{discr} K(q_p)(\pmod{(\mathbb{Z}_p)^2})$

   for all odd primes $p$ for which $l_+ + l_ - - t_+ - t_- = l((D_S)_p)$,

4. $|D_S| \equiv \pm \text{discr} K(q_2)(\pmod{(\mathbb{Z}_2)^2})$

   if $l_+ + l_ - - t_+ - t_- = l((D_S)_2)$ and $q_2 \neq \frac{x^2}{\theta^2} \oplus q'_2$ for some $\theta \in \mathbb{Z}_2^*$, and some $q'_2$.

2.2 $K3$ Surfaces

This section is devoted to recalling a few facts about $K3$ surfaces and to outlining the relevance of the theory of lattices to their study (for more details see [Bea85]).

Definition II.9. A $K3$ surface $X$ is a smooth projective complex surface with

$K_X = O_X$ and $H^1(X, O_X) = 0$.

Let $X$ be a $K3$ surface, then the Riemann-Roch theorem becomes

$$\chi(O_X(D)) = 2 + \frac{D^2}{2}.$$ 

If $X$ is a $K3$ surface, then $H^2(X, \mathbb{Z})$ is torsion free. Equipped with the cup product, $H^2(X, \mathbb{Z})$ has the structure of an even lattice. By the Hodge index theorem it has
signature \((3,19)\) and by Poincaré duality, it is unimodular (\([BHPVdV04]\)). Thus using Theorem II.6, we see that \(H^2(X,\mathbb{Z}) \cong U^3 \oplus E_8(-1)^2\).

More generally, let \(X\) be a smooth surface for which \(H^2(X,\mathbb{Z})\) is torsion free and let \(S_X\) be its Néron-Séveri group (i.e. the group of line bundles modulo algebraic equivalence). Equipped with the intersection pairing, \(S_X\) is again a lattice of signature \((1,\rho(X)-1)\), where \(\rho(X)\) is the Picard number of \(S_X\). The natural embedding \(S_X \hookrightarrow H^2(X,\mathbb{Z})\) is a primitive embedding of lattices.

**Remark II.10.** If \(X\) is a \(K3\) surface, then \(H^1(X,\mathcal{O}_X) = 0\) and thus \(S_X \cong \text{Pic}(X)\).

**Definition II.11.** Let \(X\) be a smooth surface for which \(H^2(X,\mathbb{Z})\) is torsion free. The **transcendental lattice** \(T_X\) is the orthogonal complement of \(S_X\) in \(H^2(X,\mathbb{Z})\) with respect to the cup product, i.e.

\[
T_X = S_X^\perp \subset H^2(X,\mathbb{Z}).
\]

Let \(X \to Y\) be a morphism of surfaces, the induced map on the second cohomology \(H^2(Y,\mathbb{Z}) \to H^2(X,\mathbb{Z})\) satisfies \(u^*(S_Y) \subset S_X\) and \(u^*(T_Y) \subset T_X\).

Recall that the lattice \(H^2(X,\mathbb{C})\) admits a Hodge decomposition of weight two

\[
H^2(X,\mathbb{C}) \cong H^2(X,\mathcal{O}_X) \oplus H^1(X,\Omega_X) \oplus H^0(X,\mathcal{O}_X).
\]

Similarly, the transcendental lattice \(T_X\) inherits a Hodge decomposition of weight two denoted by

\[
T_X \otimes \mathbb{C} \cong T^{2,0} \oplus T^{1,1} \oplus T^{0,2}.
\]

An isometry preserving the Hodge decomposition is called a Hodge isometry.

\(K3\) surfaces have the unusual property that they satisfy a Torelli theorem, namely

**Theorem II.12.** \([PŠŠ71]\) Let \(X\) and \(X'\) be two \(K3\) surfaces and let \(\phi : H^2(X',\mathbb{Z}) \to H^2(X,\mathbb{Z})\) be a Hodge isometry. If \(\phi\) preserves effective classes, then there exists a unique isomorphism \(u : X \to X'\) such that \(u^* = \phi\).

Mukai improved this result for \(K3\) surfaces of large Picard number and proved

**Theorem II.13.** \([Muk02]\) Let \(X\) and \(X'\) be \(K3\) surfaces and let \(\phi : T'_X \to T_X\) be a Hodge isometry. If \(\rho(X) \geq 12\), there exists an isomorphism \(u : X \to X'\) such that \(\phi = u^*\).

**Remark II.14.** The reason for the bound \(\rho(X) \geq 12\) in Theorem II.13 is because there exist \(K3\) surfaces with \(\rho(X) \leq 11\) that have Hodge isometric transcendental lattices but non isometric Néron-Séveri lattices.

**Two-Elementary Transcendental Lattices**

**Definition II.15.**

1) A lattice \(S\) is **two-elementary** if its discriminant group \(D_S \cong (\mathbb{Z}/2\mathbb{Z})^\alpha\) for some integer \(\alpha \geq 0\).

2) Let \(S\) be a two-elementary lattice. Define the invariant \(\delta_S\) to be 0 if \(q_S\) takes value in \(\mathbb{Z}/2\mathbb{Z}\), and 1 otherwise.
Theorem II.16. [Nik79] Let $S$ be an indefinite even two-elementary lattice. Then the isomorphism class of $S$ is determined by $\delta_S$, $l(D_S)$ and $\text{sign}(S)$. Moreover an even two-elementary lattice with the invariants $\delta$, $l$ and $(t_+, t_-)$ exists if and only if the following conditions are satisfied:

1) $t_+ + t_- \geq l$,
2) $t_+ + t_- + l \equiv 0 \pmod{2}$,
3) $t_+ - t_- \equiv 0 \pmod{4}$ if $\delta = 0$,
4) $\delta = 0$, $t_+ - t_- \equiv 0 \pmod{8}$ if $l = 0$,
5) $t_+ - t_- \equiv \pm 1 \pmod{8}$, if $l = 1$,
6) $\delta = 0$, if $l = 2$, $t_+ - t_- \equiv 4 \pmod{8}$,
7) $t_+ - t_- \equiv 0 \pmod{8}$ if $\delta = 0$ and $l = t_+ + t_-.$

Corollary II.17. Let $X$ be a $K3$ surface with a two-elementary transcendental lattice and Picard number 17, then $T_X \simeq U \oplus U \oplus \langle -2 \rangle$ or $T_X \simeq U(2) \oplus U \oplus \langle -2 \rangle$ or $T_X \simeq U(2) \oplus U(2) \oplus \langle -2 \rangle$.

Proof. This is a direct application of Theorem II.16. The $(2, 3)$ signature of $T_X$ implies that $\delta = 1$ $(2 - 3 \neq 0 \pmod{4})$. Moreover $l(D_T) \leq 5$ and $2 + 3 + l(D_T) \equiv 0 \pmod{2}$ imply that $l(D_T) = 1, 3$ or 5.

The geometrical meaning of two-elementary transcendental lattices is motivated by the following proposition.

Proposition II.18. [Nik79] Let $X$ be a $K3$ surface. The transcendental lattice $T_X$ is two-elementary if and only if $X$ admits an involution $\theta : X \to X$ such that $\theta^*_S = \text{id}_S$ and $\theta^*_T = -\text{id}_T$.

Example II.19. [Nik81] Let $X$ be a $K3$ surface with $T_X \simeq U \oplus U \oplus \langle -2 \rangle$. Then $\text{Aut}(X)$ is generated by two commuting involutions $\theta$ and $\iota$, where $\theta^*_S = \text{id}_S$ and $\theta^*_T = -\text{id}_T$. Moreover $X$ contains finitely many smooth rational curves which form the diagram below and where the $R_i$’s are fixed by $\theta$.

2.3 Kummer Surfaces

Let $A$ be an abelian surface, i.e. $A \simeq \mathbb{C}^2/\Lambda$ where $\Lambda$ is a free $\mathbb{Z}$-module of rank four such that $\Lambda \otimes \mathbb{R} = \mathbb{C}^2$ and $A$ is algebraic.

Definition II.20. Let $A$ denote an abelian surface and let $\iota : A \to A$ be its involution automorphism, i.e. $\iota(a) = -a$. The Kummer surface $\text{Kum}(A)$ associated to $A$ is the minimal resolution of the quotient $A/\{1, \iota\}$.

Proposition II.21. [Bea83, Nik75] The Kummer surface $\text{Kum}(A)$ is a $K3$ surface. There is a Hodge isometry $T_{\text{Kum}(A)} \simeq T_A(2)$. 

By construction, the $K3$ surface $\text{Kum}(A)$ contains sixteen disjoint smooth rational curves. Indeed the sixteen two-torsion points of $A$ are the fixed points of the involution $\iota$. The quotient $A/\{1, \iota\}$ has therefore sixteen isolated ordinary double points which give rise to sixteen disjoint rational exceptional curves on the resolution.

Conversely,

**Theorem II.22.** [Nik75] Let $Y$ be a $K3$ surface. If $Y$ contains sixteen disjoint smooth rational curves, then $Y$ is a Kummer surface.

In fact, it is possible to characterize a Kummer surface by lattice theoretical means.

**Theorem II.23.** [Nik75] There is an even, negative definite, rank sixteen lattice, $K$, called the Kummer lattice with the following properties:

1. $\text{discr}(K) = 2^6$.

2. If $Y$ is a Kummer surface, then the minimal primitive sublattice of $H^2(Y, \mathbb{Z})$ containing the classes of the exceptional curves on $Y$ is isomorphic to $K$.

3. A $K3$ surface $Y$ is a Kummer surface if and only if there is a primitive embedding $K \hookrightarrow S_Y$.

**Corollary II.24.** A general Kummer surface has its transcendental lattice isomorphic to $U(2) \oplus U(2) \oplus \langle -4 \rangle$.

**Proof.** A general abelian surface $A$ is the Jacobian of a curve of genus two $C$ and its Néron-Séveri lattice is generated by the class of $C$. By the adjunction formula $C^2 = 2$ and thus $S_A \simeq \langle 2 \rangle$. Since $H^2(A, \mathbb{Z}) \simeq U^3$, we get that $T_A \simeq U \oplus U \oplus \langle -2 \rangle$. By Proposition II.21, $T_{\text{Kum}(A)} \simeq U(2) \oplus U(2) \oplus \langle -4 \rangle$. □

Let $Y \simeq \text{Kum}(A)$ be a general Kummer surface, i.e. $A$ is the Jacobian of a curve of genus two $C$ inducing an irreducible principal polarization on $A$. The degree two
map given by the linear system \( |2C| \), \( A \rightarrow \mathbb{P}^3 \), factors through the involution \( \iota \) and hence defines an embedding \( A/\{1, \iota\} \hookrightarrow \mathbb{P}^3 \). The image of this map, \( Y_0 \), is a quartic in \( \mathbb{P}^3 \) with sixteen nodes. Denote by \( L_0 \) the class of a hyperplane section of \( Y_0 \). Projecting from a node, \( Y_0 \) admits a regular map of degree two into the projective plane which induces the map

\[
Y \stackrel{\phi}{\rightarrow} \mathbb{P}^2
\]

given by the linear system \( |L - E_0| \), where \( L \) is the pullback of \( L_0 \) on \( Y \) and \( E_0 \) is the exceptional curve resolving the center of projection. The branch locus of the map \( \phi \) is a reducible plane sextic \( S \), which is the union of six lines, \( l_1, \cdots, l_6 \), all tangent to a conic \( W \).

![Figure 2.2:](image)

Let \( p_{ij} = l_i \cap l_j \in \mathbb{P}^2 \), where \( 1 \leq i < j \leq 6 \). Index the \((3,3)\)-partitions of the set \( \{1, 2, \ldots, 6\} \), by the pair \((i,j)\) with \( 2 \leq i < j \leq 6 \). Each pair \((i,j)\) defines a plane conic \( l_{ij} \) passing through the sixtuplet \( p_{1i}, p_{1j}, p_{ij}, p_{lm}, p_{ln}, p_{mn} \), where \( \{l,m,n\} \) is the complement of \( \{1, i, j\} \) in \( \{1, 2, \ldots, 6\} \) and \( l \leq m \leq n \).

The map \( \phi \) factors as \( Y \stackrel{\psi}{\rightarrow} \tilde{\mathbb{P}}^2 \stackrel{\epsilon}{\rightarrow} \mathbb{P}^2 \) where \( \epsilon \) is the blowup of \( \mathbb{P}^2 \) at the \( p_{ij} \)’s and where \( \psi \) is the double cover of \( \tilde{\mathbb{P}}^2 \) branched along the strict transform of \( S \) in \( \tilde{\mathbb{P}}^2 \). Denote by \( E_{ij} \subset Y \) the preimage of the exceptional curves of \( \tilde{\mathbb{P}}^2 \). The ramification of the map \( \psi \) consists of the union of six disjoint smooth rational curves, \( C_0 + C_{12} + C_{13} + C_{14} + C_{15} + C_{16} \). The preimage of the ten conics \( l_{ij} \) defines ten more smooth disjoint rational curves on \( Y \), \( C_{ij}, 2 \leq i < j \leq 6 \). Finally, \( \phi(E_0) = W \).

The sixteen curves \( E_0, E_{ij}, 2 \leq i < j \leq 6 \) are called the nodes of \( Y \) and the sixteen curves \( C_0, C_{ij}, 2 \leq i < j \leq 6 \) are the tropes of \( Y \). These two sets satisfy a beautiful configuration called the \( 16_6 \) configuration, i.e. each node intersects exactly six tropes and vice versa.

**Theorem II.25.** [Hud90, Nar91] Let \( Y \) be a general Kummer surface. The Néron-Séveri lattice \( S_Y \) is generated by the classes of \( E_0, E_{ij}, C_0, C_{ij} \) and \( L \) with the relations:

1. \( C_0 = \frac{1}{2}(L - E_0 - \sum_{i=2}^{6} E_{1i}) \),
2. \( C_{1j} = \frac{1}{2}(L - E_0 - \sum_{i \neq j} E_{ij}) \),

3. \( C_{jk} = \frac{1}{2}(L - E_{1j} - E_{1k} - E_{1m} - E_{1n} - E_{mn}) \) where \( \{l, m, n\} \) are as described previously.

The intersection pairing is given by:

1. The \( E_0, E_{ij} \) are mutually orthogonal.
2. \( \langle L, L \rangle = 4, \langle L, E_0 \rangle = \langle L, E_{ij} \rangle = 0 \)
3. \( \langle E_0, E_0 \rangle = \langle E_{ij}, E_{ij} \rangle = -2 \)
4. The \( C_0, C_{ij} \) are mutually orthogonal.
5. \( \langle L, C_0 \rangle = \langle L, C_{ij} \rangle = 2 \)

The action on \( S_Y \) of the covering involution \( \alpha \) of \( Y \xrightarrow{\phi} \mathbb{P}^2 \) is given by:

\[
\begin{align*}
\alpha(C_0) &= C_0 & \alpha(C_{1j}) &= C_{1j} & 2 \leq j \leq 6 \\
\alpha(E_{ij}) &= E_{ij} & 1 \leq i < j \leq 6 & \alpha(L) &= 3L - 4E_0 \\
\alpha(E_0) &= 2L - 3E_0 & \alpha(C_{ij}) &= C_{ij} + L - 2E_0 & 2 \leq i < j \leq 6.
\end{align*}
\]

Proposition II.26. [Hud90] The minimal resolution of a double cover of \( \mathbb{P}^2 \) branched along the sextic of figure 2.2 is a Kummer surface.
CHAPTER III

K3 Surfaces associated to a Kummer Surface

3.1 Double Covers

The construction of $\text{Kum}(A)$ from an abelian surface is a particular case of a more general construction.

Let $X$ be a compact complex surface. Let $\iota$ be an involution on $X$ with isolated fixed points $Q_1, \cdots, Q_k$. Let $X \xrightarrow{\pi} X/\{1, \iota\}$ be the quotient map. The surface $X/\{1, \iota\}$ has ordinary double points at the points $P_i = \pi(Q_i)$, so that if $Y \rightarrow X/\{1, \iota\}$ is the minimal resolution of $X/\{1, \iota\}$, then the exceptional divisors of $Y$ are smooth rational curves $C_i$ of self-intersection $-2$ (i.e. $C_i$ is a nodal curve). We call the induced rational map $X \longrightarrow Y$ a rational double cover of $Y$. Alternatively, let $Z \rightarrow X$ be the blowup of the points $Q_1, \cdots, Q_k$, and let $E_i$ be the exceptional divisors of $Z$. The involution $\iota$ lifts to an involution $\hat{\iota}$ on $Z$. Taking the quotient of $Z$ by $\hat{\iota}$, we obtain the commutative diagram

```
\[
\begin{array}{ccc}
Z & \longrightarrow & X \\
\downarrow p & & \downarrow \tau \\
Y & \longrightarrow & X/\{1, \iota\}
\end{array}
\]
```

The quotient map $Z \xrightarrow{p} Y$ is a double cover branched along the smooth effective divisor $\sum_{i=1}^k C_i$.

**Lemma III.1.** Let $Y$ be a complex manifold and let $\Delta$ be a smooth effective divisor on $Y$. There exists a double cover of $Y$, $Z \xrightarrow{p} Y$ branched along $\Delta$ if and only if $\Delta \in 2\text{Pic}(Y)$.

Thus by Lemma III.1, $\sum_{i=1}^k C_i \in 2\text{Pic}(Y)$. Conversely, if $C_1, \cdots, C_k$ are disjoint nodal curves on a surface with $\sum_{i=1}^k C_i \in 2\text{Pic}(Y)$, then there is a double cover $Z \xrightarrow{p} Y$ branched along $\sum_{i=1}^k C_i$. Let $p^*(C_i) = 2E_i$, then each $E_i$ is an exceptional divisor of the first kind. We may therefore blowdown $\sum_{i=1}^k E_i$ to a smooth surface $X$ and recover a rational map of degree two $\tau : X \longrightarrow Y$. 

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**Definition III.2.** Let $Y$ be a $K3$ surface. An even set on $Y$ is a set of disjoint smooth rational curves, $C_1, \ldots, C_k$, such that
\[
\sum_{i=1}^{k} C_i \in 2S_Y.
\]
We call such a set an even eight if $k = 8$.

**Lemma III.3.** [Nik75] Let $Y$ be a $K3$ surface. If $C_1, \ldots, C_k$ is an even set on $Y$, then $k = 8$ or $16$. If $k=8$, then the surface $X$ obtained from the above construction is a $K3$ surface. If $k=16$, then $X$ is an abelian surface.

From the above lemma, it becomes apparent that a Kummer surface $Y$ can be associated not only to an abelian surface but also, via even eights, to a $K3$ surface. Moreover such a $K3$ surface must admit an involution with eight isolated fixed points.

### 3.2 Symplectic Involutions

**Definition III.4.** Let $X$ be a $K3$ surface, an involution $X \xrightarrow{\iota} X$ is symplectic if $\iota^*(\omega) = \omega$ for all $\omega \in H^{0,2}(X)$.

**Proposition III.5.** [Nik91] 1) Every symplectic involution has eight isolated fixed points.

2) If $X \xrightarrow{\tau} Y$ is a rational quotient of a $K3$ surface by a symplectic involution, then $Y$ is a $K3$ surface.

3) Conversely, if $X \xrightarrow{\tau} Y$ is a rational map of degree two between $K3$ surfaces, then the covering involution $\iota : X \rightarrow X$ is a symplectic involution.

Let $X$ be a $K3$ surface with a symplectic involution $\iota$. The two lattices
\[
T^\iota = \{ x \in H^2(X, \mathbb{Z}) | \iota^*(x) = x \} \quad \text{and} \quad S_\iota = \{ x \in H^2(X, \mathbb{Z}) | \iota^*(x) = -x \}
\]
define the primitive embeddings
\[
T_X \hookrightarrow T^\iota \hookrightarrow H^2(X, \mathbb{Z}), \quad S_\iota \hookrightarrow S_X \hookrightarrow H^2(X, \mathbb{Z}).
\]
Nikulin showed that
\[
T^\iota \simeq U^3 \oplus E_8(-2) \quad \text{and} \quad S_\iota \simeq E_8(-2).
\]

**Definition III.6.** The Nikulin lattice is an even lattice $N$ of rank eight generated by $\{c_i\}_{i=1}^8$ and $d = \frac{1}{2} \sum_{i=1}^8 c_i$, with the bilinear form $c_i \cdot c_j = -2\delta_{ij}$. 

It follows from the discussion in section 3.1 that if $X \to Y$ is a rational map of degree two between $K3$ surfaces, then there exists an embedding of the Nikulin lattice into $S_Y$. In fact this embedding is primitive. Indeed denote by $N'$ the sublattice of $S_Y$ generated by $C_1, \cdots, C_8$, the class of the exceptional divisors of $\tau$ and assume that there exists $C = \sum m_i C_i \in S_Y$ with $m_i \in \mathbb{Q}$. Then $C \cdot C_j = -2m_j \in \mathbb{Z}$ and consequently $C - \frac{1}{2} \sum_{i \in I} C_i \in N'$ for some $I \subset \{1, \cdots, 8\}$. But by Lemma III.3 $I = \{1, \cdots, 8\}$. Thus $C_1, \cdots, C_8$ and $\frac{1}{2} \sum_{i=1}^{8} C_i$ define a primitive embedding of the Nikulin lattice into $S_Y([Mor84])$.

Moreover, the induced maps on the transcendental lattices

$$\tau^* : T_Y \to T_X \quad \text{and} \quad \tau_* : T_X \to T_Y$$

preserve the Hodge decomposition and satisfy

$$\tau_* \tau^*(x) = 2x, \quad \tau^* \tau_*(x) = 2x, \quad \tau^*(x) \cdot \tau^*(y) = 2xy, \quad \tau_*(x) \cdot \tau_*(y) = 2xy.$$ 

The image of $T_Y$ by $\tau^*$ is therefore a sublattice of $T_X$ isomorphic to $T_Y(2)$ with two-elementary quotient, i.e

$$T_X / T_Y(2) \cong (\mathbb{Z}/2\mathbb{Z})^\beta \text{ for some } \beta \geq 0.$$ 

**Lemma III.7.** ([Nik75]) Let $X$ be a $K3$ surface and let $T_X \to T \cong U^3 \oplus E_8(-2)$ be a primitive embedding of lattices. Then there exists a symplectic involution $\iota$ of $X$ such that for the corresponding rational map of degree two $X \to Y$

$$\tau^* T_Y \cong T_Y(2) \cong 2(T^* \cap (T_X \otimes \mathbb{Q})) \subset T_X.$$ 

**Connection to lattices.** All the results of this section can be found in [Nik91]. Let $L$ be an unimodular lattice and let $T$ and $S$ be two primitive sublattices of $L$ which are orthogonal to each other. Let $M$ be the primitive sublattice of $L$ generated by $T \oplus S$. Then the subgroup $\Gamma = M / (T \oplus S) \subset D_T \oplus D_S$ is isotropic and satisfies

$$\Gamma \cap (D_T \oplus \{0\}) = 0 \quad \text{and} \quad \Gamma \cap (\{0\} \oplus D_S) = 0.$$ 

Let $\pi_T$ and $\pi_S$ be the projections in $D_T$ and $D_S$ respectively. Let $\mathfrak{H} = \pi_T(\Gamma) \subset D_T$. Note that $\mathfrak{H}$ is not necessarily isotropic. The map $\xi = \pi_S \circ \pi_T^{-1}$ defines an inclusion of quadratic forms (i.e. an injection of groups preserving the bilinear form)

$$\xi : \mathfrak{H} \to (D_S, -q_S).$$

**Lemma III.8.** ([Nik91]) $((S^\perp)^* \cap (T \otimes \mathbb{Q}))/T = \mathfrak{H} \subset D_T$.

In particular if $S \cong E_8(-2)$ and $\alpha = l(\mathfrak{H}) \leq 4$, then $D_M \cong D_T \oplus u(2)^{1-\alpha}$, where $u(2)$ denotes the discriminant form of the lattice $U(2)$. 

---

**Correction Note:** The text contains references to Lemmas III.3 and III.7, which are likely intended to be from the cited sources (Nikulin lattices and their embeddings). The specific details and proofs of these lemmas are part of a larger theoretical framework and are assumed to be known or derivable from the cited references. The focus here is on the narrative and application of these concepts within the context of rational maps between $K3$ surfaces and their associated lattices.
3.3 \( K3 \) Surfaces Covering a Kummer Surface

We can now give necessary and sufficient conditions for a \( K3 \) surface to admit a symplectic involution such that the quotient is birational to a fixed Kummer surface.

**Theorem III.9.** Let \( Y \simeq \text{Kum}(A) \) be a Kummer surface and let \( X \) be a \( K3 \) surface. \( X \) admits a rational map of degree two \( X \dashrightarrow Y \) if and only if there exists an embedding of lattices \( \phi : T_X \hookrightarrow T_A \) satisfying

1. \( \phi \) preserves the Hodge decomposition,
2. \( T_A/T_X \simeq (\mathbb{Z}/2)^\alpha \) with \( 0 \leq \alpha \leq 4 \),
3. \( |D_{T_X}| \equiv \pm \text{discr}K(q_{T_X})_2 \mod (\mathbb{Z}_2^*)^2 \),
   \( \alpha = \frac{\text{rank}(T_X)+l((D_{T_X})_x)}{2} - 3 \) and \( (q_{T_X})_2 \oplus u^{1-\alpha}(2) \not\equiv \frac{x^2}{\theta^2} \oplus q_2' \) for some \( \theta \in \mathbb{Z}_2^* \) and some \( q'_2 \).

**Proof.** Suppose \( X \) admits a rational map of degree two \( X \dashrightarrow Y \). The covering involution \( \iota : X \to X \) is then symplectic. Consider the induced Hodge isometry of Lemma III.7

\[ T_Y(2) \cong 2(T_X \otimes \mathbb{Q} \cap (T^*)^*) \subset T_X. \]

Denote by \( \tilde{\mathfrak{H}} \) the rational overlattice of \( T_X \) defined by \( T_X \otimes \mathbb{Q} \cap (T^*)^* \). Since \( 2\tilde{\mathfrak{H}} \subset T_X \), the quotient \( T_X/\tilde{\mathfrak{H}} \) is two-elementary, i.e. \( T_X/\tilde{\mathfrak{H}} \simeq (\mathbb{Z}/2)^\alpha \) for some \( \alpha \geq 0 \).

Recall from Proposition II.21 that \( T_A(2) \) is Hodge isometric to \( T_Y \) and consequently \( T_A(4) \) is Hodge isometric to \( T_Y(2) \). Thus there is a Hodge isometry

\[ T_A(4) \cong 2\tilde{\mathfrak{H}} \subset T_X. \]

The obvious identification of \( T_A(4) \) with \( 2T_A \) shows that \( T_A \simeq \tilde{\mathfrak{H}} \) and that the inclusion preserving the Hodge decomposition \( T_X \hookrightarrow T_A \) satisfies

\[ T_A/T_X \simeq \tilde{\mathfrak{H}}/T_X \simeq (\mathbb{Z}/2)^\alpha \] for some \( \alpha \geq 0 \).

The lattice \( \tilde{\mathfrak{H}} \) is therefore an integral overlattice of \( T_X \) and the group \( \tilde{\mathfrak{H}}/T_X \) is an isotropic subgroup of \( (D_{T_X}, q_{T_X}) \). By setting \( T = T_X \) and \( S = S_i \) in lemma III.8, we see that the group \( \tilde{\mathfrak{H}}/T_X \) must be isotropic. Since \( \tilde{\mathfrak{H}} \) admits an inclusion of quadratic forms into \( (D_{S_i}, -q_{S_i}) \), where \( l(D_{S_i}) = 8 \), we obtain the upper bound \( \alpha \leq 4 \).

Denote by \( M \) the primitive closure of \( T_X \oplus S_i \) in \( H^2(X, \mathbb{Z}) \), then \( D_M \simeq D_{T_X} \oplus u(2)^{4-\alpha} \). The lattice \( M \) admits a primitive embedding into the unimodular lattice in \( H^2(X, \mathbb{Z}) \), so it satisfies the hypothesis of Theorem II.8. In particular, it satisfies

\[ |D_{T_X}| \equiv \pm \text{discr}K(q_{T_X})_2 \mod (\mathbb{Z}_2^*)^2 \]

if \( \alpha = \frac{\text{rank}(T_X)+l((D_{T_X})_x)}{2} - 3 \) and \( (q_{T_X})_2 \oplus u^{1-\alpha}(2) \not\equiv \frac{x^2}{\theta^2} \oplus q_2' \) for some \( \theta \in \mathbb{Z}_2^* \).
Conversely, assume that $X$ satisfies conditions 1, 2 and 3.  

**Claim:** There exists a primitive embedding $T_X \hookrightarrow U^3 \oplus E_8(-2)$ such that 

$$(T_X \otimes \mathbb{Q}) \cap (U^3 \oplus E_8(-2)^*) \simeq T_A.$$ 

Assuming the claim, by Lemma III.7, there exists a $K3$ surface $\bar{Y}$ and a symplectic involution on $X$ such that the corresponding rational map of degree two $X \dashrightarrow \bar{Y}$ satisfies 

$$T\bar{Y}(2) \simeq 2(U^3 \oplus E_8(-2) \cap T_X \otimes \mathbb{Q}) \subset T_X.$$ 

The Hodge isometry $T\bar{Y}(2) \simeq 2T_A$ implies that $T\bar{Y} \simeq T_A(2)$ and by Theorem II.13, we get that $\bar{Y} \simeq Y$.

**Proof of the claim:**

Let $\mathfrak{H} \simeq \mathbb{Z}/2^\alpha$ be the isotropic subgroup of $D_{T_X}$ corresponding to the overlattice $T_X \subset T_A$. Since $\alpha \leq 4$, there exists an embedding of quadratic forms $\mathfrak{H} \hookrightarrow (D_S, -q_S)$, where $S \simeq E_8(-2)$. Consider the subgroup $\Gamma_\xi = \{h + \xi(h) | h \in \mathfrak{H}\} \subset D_{T_X} \oplus D_S$. Clearly $\Gamma_\xi$ is an isotropic subgroup of $D_{T_X} \oplus D_S$. Denote by $M$ the overlattice it defines. Observe that

$$\text{rank}_T M = \text{rank}_T X + 8 \text{ and } D_M \simeq q_{T_X} \oplus u(2)^{4-\alpha}.$$ 

For $M$ to admit a primitive embedding into $L = H^2(X, \mathbb{Z})$, it must satisfy the hypothesis of Theorem II.8 which in our setting involves checking that

$$l((D_M)_p) + \text{rank}(T_X) \leq 14 \quad \forall p \text{ prime}.$$ 

Note that for any finite abelian group, $l(D) = \max_p l(D_p)$, so

$$l(D_M) + \text{rank}(M) \leq \text{rank}(L) \iff l((D_M)_p) + \text{rank}(M) \leq \text{rank}(L) \quad \forall p \text{ prime}.$$ 

If $p$ is odd, then $(D_M)_p \simeq (D_{T_X})_p$ and the strict inequality $l((D_M)_p) + \text{rank}(T_X) < 14$ always holds for $l(D_{T_X}) \leq \text{rank}T_X \leq 5$.

If $p$ is 2, then $l((D_M)_2) = l((D_{T_X})_2) + 8 - 2\alpha$. The facts that $l((D_{T_X})_2) - 2\alpha \leq l((D_A)_2)$, which is proved in the technical lemma III.10 below, and the lattice $T_A$ embeds primitively into $U^3$ imply that

$$\text{rank}(T_X) + l((D_{T_X})_2) - 2\alpha \leq \text{rank}(T_A) + l((D_{T_A})_2) \leq \text{rank}(T_A) + l(D_{T_A}) \leq 6.$$ 

Thus the inequality $l(D_M) + \text{rank}(M) \leq 22$ always holds and the assumption that

$$|D_{T_X}| \equiv \pm \text{discr} K(q_{T_X})_2 \text{ mod } (\mathbb{Z}_2)^2$$ 

if $\alpha = \frac{\text{rank}T_X + l(D_{T_X})_2}{2} - 3$ and $(q_{T_X})_2 \oplus u^{4-\alpha}(2) \not\equiv \frac{x^2}{2} \oplus q_2'$ for some $\theta \in \mathbb{Z}_2^*$, is enough to guarantee the existence of the primitive embedding of $M$ into $H^2(X, \mathbb{Z})$, i.e.

$$T_X \oplus S \subset M \subset H^2(X, \mathbb{Z}).$$
The restriction of this embedding to $T_X$ and to $S$ is primitive. It follows that $S$ is primitively embedded in $S_X$ or equivalently $T_X$ is primitively embedded in $S^\perp \simeq U^3 \oplus E_8(-2)$. Finally, by Lemma III.8, we see that

$$\mathcal{F} \simeq (T_X \otimes \mathbb{Q}) \cap (U^3 \oplus E_8(-2)^*)/T_X$$

and therefore

$$(T_X \otimes \mathbb{Q}) \cap (U^3 \oplus E_8(-2)^*) \simeq T_A.$$

Lemma III.10. Let $D$ be a finite abelian 2-group with a bilinear form, $b(x, y) \in \mathbb{Q}/\mathbb{Z}$ and $q(x) = b(x, x) \in \mathbb{Q}/2\mathbb{Z}$. Let $\mathcal{F} \subset D$ be an isotropic subgroup, then

$$l(D) - 2l(\mathcal{F}) \leq l(\mathcal{F}^\perp/\mathcal{F}).$$

Proof. It is enough to show that $l(D/\mathcal{F}) \leq l(\mathcal{F}^\perp)$. Indeed it would imply that

$$l(D) \leq l(D/\mathcal{F}) + l(\mathcal{F}) \leq l(\mathcal{F}^\perp) + l(\mathcal{F}) \leq l(\mathcal{F}^\perp/\mathcal{F}) + l(\mathcal{F}) + l(\mathcal{F})$$

where the first and last inequalities are obvious.

Assume first that $\mathcal{F} \simeq \mathbb{Z}/2$ with generator $h$. Let $[x] \in D/\mathcal{F}$ be a non-zero element of order $2^r$, then

$$q(h) = b(h, h) = b(2^rx, h) = 2b(2^{r-1}x, h) = 0 \Rightarrow b(2^{r-1}x, h) = 0 \text{ therefore } 2^{r-1}x \in \mathcal{F}^\perp.$$

Under this correspondence, a minimal set of generators of $D/\mathcal{F}$ maps to a set of independent elements of $\mathcal{F}^\perp$, i.e. $l(D/\mathcal{F}) \leq l(\mathcal{F}^\perp)$.

If $|\mathcal{F}| > 1$, then there exists a subgroup $\mathcal{F}' \subset \mathcal{F}$, such that $\mathcal{F}/\mathcal{F}' \simeq \mathbb{Z}/2$. Clearly $\mathcal{F}/\mathcal{F}'$ is an isotropic subgroup of $D_S/\mathcal{F}'$. Again, a minimal set of generators of $(\mathcal{F}/\mathcal{F}')^\perp$ maps to a set of independent elements of $\mathcal{F}^\perp$, thus $l(\mathcal{F}/\mathcal{F}') \leq l(\mathcal{F}^\perp)$. By induction we get that

$$l(\mathcal{F}^\perp) \geq l((\mathcal{F}/\mathcal{F}')^\perp \geq l((D/\mathcal{F}))/\mathcal{F}/\mathcal{F}') = l(D/\mathcal{F}).$$

Corollary III.11. Let $Y$ be a Kummer surface. There exist finitely many isomorphism classes of K3 surfaces admitting a rational map of degree two $X \to Y$

Proof. Any K3 surface with $X \to Y$ has Picard number greater or equal to 17, thus by Theorem II.13, it is determined by the Hodge decomposition of its transcendental lattice. But $T_A$ admits finitely many sublattices of a given index.

Let $S$ be a sublattice of $T$. Note that

$$T/S \simeq (\mathbb{Z}/2)^\alpha \hookrightarrow S \text{ overlattice of } 2T \leftrightarrow 2T/S \text{ isotropic subgroup of } D_{2T}.$$
Let $Y$ be a general Kummer surface and $T_Y \simeq U(2) \oplus U(2) \oplus (-4)$.
Set $T_0 = U \oplus U \oplus (-2)$. The discriminant form $(D_{2T_0}, q)$ is given by
\[
\mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/8, \quad q(x) = \frac{1}{2}x_1x_2 + \frac{1}{2}x_3x_4 - \frac{1}{8}x_5^2.
\]

Using Proposition II.5 and a computer program, we compute all the overlattices of $2T_0$. According to Remark II.3, any such lattice is determined by its discriminant form. Therefore, we obtain a short and exhaustive list of all possible isomorphism classes of lattices satisfying the conditions of Theorem III.9 when $T_A \simeq U \oplus U \oplus (-2)$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$T_X$</th>
<th>$D_{T_X}$</th>
<th>$q_{T_X}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$U \oplus U \oplus (-2)$</td>
<td>$\mathbb{Z}/2\mathbb{Z}$</td>
<td>$\frac{1}{2}x_1^2$</td>
</tr>
<tr>
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<td>$U(2) \oplus U \oplus (-2)$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$</td>
<td>$x_1x_2 - \frac{1}{8}x_3^2$</td>
</tr>
<tr>
<td>1</td>
<td>$U \oplus U \oplus (-8)$</td>
<td>$\mathbb{Z}/8\mathbb{Z}$</td>
<td>$-\frac{1}{8}x_3^2$</td>
</tr>
<tr>
<td>1</td>
<td>$N_1$</td>
<td>$\mathbb{Z}/8\mathbb{Z}$</td>
<td>$\frac{1}{8}x_3^2$</td>
</tr>
<tr>
<td>2</td>
<td>$U(4) \oplus U \oplus (-2)$</td>
<td>$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$</td>
<td>$\frac{1}{2}x_1x_2 - \frac{1}{8}x_3^2$</td>
</tr>
<tr>
<td>2</td>
<td>$U(2) \oplus U(2) \oplus (-2)$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$</td>
<td>$x_1x_2 + \frac{3}{8}x_3^2$</td>
</tr>
<tr>
<td>2</td>
<td>$U(2) \oplus U \oplus (-8)$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$</td>
<td>$-x_1x_2 + \frac{3}{8}x_3^2$</td>
</tr>
<tr>
<td>2</td>
<td>$N_2$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$</td>
<td>$\frac{1}{2}x_1x_2 - \frac{1}{8}x_3^2$</td>
</tr>
<tr>
<td>2</td>
<td>$N_3$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$</td>
<td>$-x_1x_2 + \frac{3}{8}x_3^2$</td>
</tr>
<tr>
<td>3</td>
<td>$U(4) \oplus U(2) \oplus (-2)$</td>
<td>$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$</td>
<td>$\frac{1}{2}x_1x_2 - \frac{1}{8}x_3^2$</td>
</tr>
<tr>
<td>3</td>
<td>$U(4) \oplus U \oplus (-8)$</td>
<td>$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$</td>
<td>$\frac{1}{2}x_1x_2 - \frac{1}{8}x_3^2$</td>
</tr>
<tr>
<td>3</td>
<td>$N_4$</td>
<td>$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$</td>
<td>$\frac{1}{2}x_1x_2 + \frac{3}{8}x_3^2$</td>
</tr>
<tr>
<td>3</td>
<td>$U(2) \oplus U(2) \oplus (-8)$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$</td>
<td>$x_1x_2 + \frac{3}{8}x_3^2$</td>
</tr>
<tr>
<td>3</td>
<td>$N_5$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$</td>
<td>$\frac{1}{2}x_1x_2 - \frac{1}{8}x_3^2$</td>
</tr>
<tr>
<td>4</td>
<td>$U(4) \oplus U(4) \oplus (-2)$</td>
<td>$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$</td>
<td>$\frac{1}{2}x_1x_2 - \frac{1}{8}x_3^2$</td>
</tr>
<tr>
<td>4</td>
<td>$U(4) \oplus U(2) \oplus (-8)$</td>
<td>$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$</td>
<td>$x_1x_2 + \frac{3}{8}x_3^2$</td>
</tr>
<tr>
<td>4</td>
<td>$U(4) \oplus U(4) \oplus (-8)$</td>
<td>$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$</td>
<td>$\frac{1}{2}x_1x_2 - \frac{1}{8}x_3^2$</td>
</tr>
</tbody>
</table>

The lattices $N_1, \ldots, N_5$, are given by the symmetric integral matrices

\[
N_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -2 & -1 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}, \quad
N_2 = \begin{pmatrix}
-2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 2 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad
N_3 = \begin{pmatrix}
0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & -1 \\
1 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 2 \\
0 & 0 & -1 & -1 & 2
\end{pmatrix},
\]

\[
N_4 = \begin{pmatrix}
2 & 0 & -2 & 2 & 0 \\
0 & -2 & 0 & 2 & 0 \\
-2 & 0 & 2 & 0 & 2 \\
2 & 2 & 0 & -2 & 2 \\
0 & 0 & 2 & 2 & -2
\end{pmatrix}, \quad
N_5 = \begin{pmatrix}
-2 & 2 & -1 & 3 & 0 \\
2 & -2 & -1 & -1 & 0 \\
-1 & 1 & -2 & 1 & 2 \\
3 & -1 & 1 & -2 & 2 \\
0 & 0 & 2 & 2 & 0
\end{pmatrix}.
\]

### 3.4 Shioda-Inose Structure

A particular case in this list is given by the $K3$ surface $X$ with $T_X \simeq U \oplus U \oplus (-2)$. Such a $K3$ surface is said to have a Shioda-Inose structure.
**Definition III.12.** A K3 surface $X$ has a Shioda-Inose structure if there is a symplectic involution $\iota$ on $X$ with rational quotient map $X \xrightarrow{\tau} Y$, such that $Y$ is a Kummer surface and $\tau_*$ induces a Hodge isometry $T_X(2) \cong T_Y$.

Morrisson proved in [Mor84] that if $X$ is a K3 surface with a Shioda-Inose structure, and $X \xrightarrow{\tau} Y$ is the rational quotient map by the symplectic involution, then there is an embedding of lattices $E_8(-1) \hookrightarrow N^\perp \subset S_Y$, where $N$ is the Nikulin lattice defined in section 3.2.

When $Y$ is a general Kummer surface this can be seen by considering the following $(-2)$-classes on $Y$.

\[
\begin{align*}
    r_1 & := C_{14} & r_2 & := E_{14} & r_3 & := E_{15} & r_4 & := C_0 \\
    r_5 & := E_{16} & r_6 & := C_{16} & r_7 & := E_{26} & r_8 & := C_{12} \\
    e_1 & := (L - E_0) - (E_{12} + E_{46}) \\
    e_2 & := 2(L - E_0) - (E_{12} + E_{24} + E_{46} + E_{56}) \\
    e_3 & := 3(L - E_0) - 2E_{12} - (E_{13} + E_{24} + E_{36} + E_{45} + E_{56}) \\
    e_4 & := 4(L - E_0) - 2(E_{12} + E_{24} + E_{46}) - (E_{24} + E_{25} + E_{36} + E_{45} + E_{56}) \\
    e_5 & := 5(L - E_0) - 3E_{12} - 2(E_{13} + E_{46} + E_{56}) + (E_{24} + E_{25} + E_{34} + E_{36} + E_{45}) \\
    e_6 & := C_{23} & e_7 & := \alpha(C_{23}) & e_8 & := E_{35}.
\end{align*}
\]

**Theorem III.13.** [Nar91] The sixteen $(-2)$-classes, $r_1, \ldots, r_8, e_1, \ldots, e_8$ are represented by smooth rational curves. They generate a copy of the lattice $E_8(-1) \oplus N$ in $S_Y$.

Note that

\[
\sum_{i=1}^8 e_i = 2C_{13} + 2\{8(L - E_0) - (5E_{12} + 3E_{13} + 2E_{24} + E_{25} + E_{35} + 2E_{45} + 4E_{46} + 3E_{56})\} \in 2S_Y.
\]

The geometric meaning of the $(-2)$-classes can be explained in terms of the double plane model of the general Kummer surface $Y \xrightarrow{\phi} \mathbb{P}^2$ described in section 2.3. The class $e_1$ is represented by the proper inverse image of a line passing through $p_{12}$ and $p_{46}$. The class of $e_2$ is represented by the proper inverse image of a conic passing through $p_{12}, p_{13}, p_{24}, p_{46}, p_{56}$. The class of $e_3$ is represented by the proper inverse image of a cubic passing through $p_{13}, p_{24}, p_{36}, p_{45}, p_{46}, p_{56}$ and having a double point at $p_{12}$. The class of $e_4$ is represented by the proper inverse image of a quartic passing through $p_{24}, p_{25}, p_{36}, p_{45}, p_{56}$ and having double points at $p_{12}, p_{13}, p_{46}$. The class of $e_5$ is represented by the proper inverse image of a quintic passing through $p_{24}, p_{25}, p_{34}, p_{36}, p_{45}$ and having double points at $p_{13}, p_{46}, p_{56}$ and a triple point at $p_{12}$.

Moreover, there exists a reducible plane sextic $S$, decomposable as a cubic $C$ with a cusp, a line $L$ intersecting the cubic only at the cusp and a conic $Q$ tangent to the line, (see figure 3.1) such that the map $X \xrightarrow{\tau} Y$, decomposes as
where \( \pi, \pi_1 \) and \( \pi_2 \) are double covers branched along \( S, Q \) and \( \pi_1^*(C + L) \) respectively ([Nar91], [GL04]).

![Diagram](image)

Figure 3.1:
CHAPTER IV

Two-Elementary Transcendental Lattices

4.1 Basics about Elliptic Fibrations

Definition IV.1. A smooth surface $X$ is elliptic if it admits a regular map $X \to C$ where the general fiber is a smooth connected curve of genus one. When nonempty, the set of sections of $f$, $MW_f(X)$, form a finitely generated group called the Mordell-Weil group, which can be identified with the set of $k(C)$-rational points of the generic fiber $X_\eta$.

Theorem IV.2. [Shi90] Let $X \to \mathbb{P}^1$ be an elliptic $K3$ surface with a section $S_0$. Let $A$ be the subgroup of $S_X$ generated by the section $S_0$, the class of a fiber and the components of the reducible fibers that do not meet $S_0$. There is a short exact sequence of groups

$$0 \to A \xrightarrow{\alpha} S_X \xrightarrow{\beta} MW_f(X) \to 0$$

where $\beta$ is the restriction to the generic fiber. Let $\Theta_i (i = 1, \ldots, k)$ be all the singular fibers of $f$. Then

1. $24 = \chi_{top}(X) = \sum_i \chi_{top}(\Theta_i)$,
2. $\text{rank} MW_f(X) = \text{rank} S_X - 2 - \sum_i (m(\Theta_i) - 1)$, where $m(\Theta_i)$ denotes the number of irreducible components of $\Theta_i$.
3. When the rank of $MW_f(X)$ is zero, then

$$|D_{S_X}| = \frac{\prod_i s_i}{n^2}$$

where $s_i$ is the number of simple components of the singular fiber $\Theta_i$ and $n$ is the order of the finite group $MW_f(X)$.

Pjateckii-Šapiro and Šafarevič showed that on a $K3$ surface, if $D$ is a non-zero effective nef divisor with $D^2 = 0$, then the system $|D|$ is of the form $|mD_0|, m \geq 1$ and the linear system $|D_0|$ defines a morphism $X \to \mathbb{P}^1$, with general fiber a smooth connected curve of genus one ([PSS71]).
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Structure</th>
<th>Number of components</th>
<th>Euler Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_0$</td>
<td>non singular curve of genus one</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$I_1$</td>
<td>nodal curve</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$II$</td>
<td>cuspidal curve</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$I_2$</td>
<td></td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$III$</td>
<td></td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$IV$</td>
<td></td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$I_b$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$I_{b*}$</td>
<td></td>
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<tr>
<td>$II^*$</td>
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</tr>
<tr>
<td>$III^*$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$IV^*$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.1: All possible non-multiple singular fibers on a smooth minimal elliptic surface
**Definition IV.3.** Let $X \stackrel{f}{\to} \mathbb{P}^1$ be an elliptic K3 surface and $F$ a general fiber of $f$. The set

$$\{ C \cdot F \mid C \in S_X \}$$

is an ideal of $\mathbb{Z}$. The multisection index $l$ of $f$ is the positive generator of this ideal. An $l$—section is a divisor $C$ on $X$ for which $C \cdot F = l$.

**Proposition IV.4.** Let $Y$ be a general Kummer surface. Then there exists an elliptic fibration on $Y$ whose singular fibers are of the type $6I_2 + I_5^* + I_1$ and whose Mordell-Weil group is cyclic of order two.

**Proof.** Consider the divisor class

$$D = 5(L - E_0) - 3E_{12} - 2(E_{13} + E_{46} + E_{56}) - (E_{24} + E_{25} + E_{36} + E_{45}).$$

Geometrically, $D$ can be represented by the proper inverse image, under the map $Y \xrightarrow{\phi} \mathbb{P}^2$, of a quintic passing through $p_{24}$, $p_{25}$, $p_{36}$, $p_{45}$, having double points at $p_{13}$, $p_{46}$, $p_{56}$ and a triple point at $p_{12}$. Using the $e_i$'s defined in section 3.4, we see that $D \sim e_5 + E_{34}$ and $e_5 \cdot E_{34} = 2$. Hence $D$ is a reduced nef effective divisor with $D^2 = 0$. Let $Y \xrightarrow{f} \mathbb{P}^1$ be the elliptic fibration defined by the linear system $|D|$. The divisors

1. $F_1 = e_5 + E_{34}$
2. $F_2 = e_4 + (L - E_0) - (E_{12} + E_{56})$ 
3. $F_3 = e_3 + 2(L - E_0) - (E_{12} + E_{13} + E_{46} + E_{56} + E_{25})$
4. $F_4 = e_2 + 3(L - E_0) - 2E_{12} - (E_{13} + E_{46} + E_{56} + E_{25} + E_{36} + E_{45})$
5. $F_5 = e_1 + 4(L - E_0) - 2(E_{12} + E_{13} + E_{56}) - (E_{46} + E_{24} + E_{25} + E_{36} + E_{45})$
6. $F_6 = e_8 + 5(L - E_0) - 3E_{12} - 2(E_{13} + E_{46} + E_{56}) - (E_{24} + E_{25} + E_{36} + E_{45} + E_{35})$

represent six fibers of type $I_2$. Under the genericity assumption, the $(-2)$-cycles, $a_1, \ldots, a_5$ are represented by smooth rational curves. Their geometric meaning can be explained exactly in the same way as in Theorem III.13. The divisor

$$F_7 = e_6 + e_7 + 2(E_{23} + C_{12} + E_{26} + C_{16} + E_{16} + C_0) + E_{14} + E_{15}$$

defines a fiber of type $I_5^*$. Note also that $C_{14} \cdot D = C_{15} \cdot D = 1$, i.e. the curves $C_{14}$ and $C_{15}$ are two sections of $f$. The fifteen components of the singular fibers,

$$e_1, \ldots, e_8, C_{12}, E_{23}, E_{26}, C_{16}, E_{16}, C_0, E_{14}$$

are independent over $\mathbb{Q}$. Consequently, by Theorem IV.2, there are no other reducible fibers and the Mordell-Weil group is finite of order two. \qed
4.2 $K3$ Surfaces with $T_X \simeq U(2) \oplus U \oplus \langle -2 \rangle$

In this section, we give a new even eight on a general Kummer surface and show that the corresponding rational double cover $X$ has its transcendental lattice isomorphic to $U(2) \oplus U \oplus \langle -2 \rangle$. Moreover we exhibit an elliptic fibration on $X$ such that the symplectic involution is obtained as the composition of the natural involution on the general fiber with an involution acting non trivially on the base. We will keep the same notation as in the previous section.

Set $a_6 := e_6, \quad a_7 := e_7, \quad a_8 := E_{34}$.

**Lemma IV.5.** The $a_i$’s form an even eight on $Y$. If $X \xrightarrow{\tau} Y$ is the associated rational map of degree two, then $T_X \simeq U(2) \oplus U \oplus \langle -2 \rangle$.

**Proof.** A direct computation shows that
\[
\sum_{i=1}^{8} a_i = 2C_{13} + 2E_{34} + 2 \cdot \{8(L-E_0) - (5E_{12} + 4E_{46} + 3E_{13} + 3E_{56} + E_{36} + E_{25} + E_{45})\} \in 2S_Y.
\]

Recall that the associated rational double cover $X \xrightarrow{\tau} Y$ factors as
\[
\begin{array}{ccc}
Z & \xrightarrow{\epsilon} & X \\
\downarrow p & & \downarrow \tau \\
Y & & \end{array}
\]

where $Z \xrightarrow{p} Y$ is a degree two map branched along the $a_i$’s and $Z \xrightarrow{\epsilon} X$ contracts the eight exceptional curves $p^{-1}(a_i)$. Let $X \xrightarrow{\iota} X$ be the covering symplectic involution. We will describe the proper inverse image of the elliptic fibration of Proposition IV.4 under the map $\tau$. Each of the curves $e_1, e_2, e_3, e_4, e_5, e_8$ meet the $a_i$’s at exactly two points, thus their pullback under $p$ are smooth rational curves each meeting an exceptional curve at two points. After blowing down the exceptional curves, they become six disjoint nodal curves $f_1, f_2, f_3, f_4, f_5, f_8$. It follows from the properties of $\tau^*$ introduced in section 3.2 that the divisor $\tau^*D$ is a nonzero effective nef divisor and has self-intersection 0. Thus the linear system $|\tau^*D|$ is of the form $|mD_0|, m \geq 1$, where $D_0$ is a smooth curve of genus one. Clearly the $f_i$’s are singular fibers of the elliptic fibration defined by $|D_0|, X \xrightarrow{g} \mathbb{P}^1$. Moreover as $\iota(f_i) = f_i$, the involution $\iota$ must act trivially on $\mathbb{P}^1$ under the map $g$.

**Claim:** The general member of the linear system $|\tau^*D|$ is a smooth curve of genus one, i.e. $\tau^*D \sim D_0$.

**Proof of the claim:** First note that since $mD_0 \cdot \tau^*C_{14} = \tau^*D \cdot \tau^*C_{14} = 2D \cdot C_{14} = 2$, we must have $m \leq 2$. Suppose that $m = 2$, then it would mean that the preimage of the general member of $|D|$ splits into two disjoint smooth curves of genus one
\[
\tau^{-1}(D) = D_0 + D_1, \quad D_0 \sim D_1,
\]
then \( \iota \) maps \( D_0 \) to \( D_1 \), which contradicts the assumption that \( \iota \) acts trivially on \( \mathbb{P}^1 \).

We can infer from the claim that the preimage of the \( I_1 \) fiber on \( Y \) does not split into two \( I_1 \) fibers but becomes a singular fiber of type \( I_2 \). Denote its two components by \( S \) and \( \iota^* S \). Finally, by Hurwitz formula the \( I_5^* \) fiber pulls back to a \( I_{10}^* \) fiber of \( g \) whose components we will denote by

\[
E'_{14} + E'_{15} + 2(C'_{0} + E'_{16} + C'_{16} + E'_{26} + C'_{12} + E'_{23} + \iota^* C'_{12} + \iota^* E'_{26} + \iota^* C'_{16} + \iota^* E'_{16} + \iota^* C'_{0}) + \iota^* E'_{14} + \iota^* E'_{15}.
\]

The section \( C_{14} \) meets the even eight at two points so it pulls back to a two-section \( C'_{14} \). We conclude that \( X \) contains the following diagram of \((-2)\) curves

![Diagram of curves](image)

Figure 4.1:

We also observe that the curves \( C_{13} \) and \( C_{15} \) pull back to a curve of genus two \( W \) and a curve of genus one \( E \), respectively, both invariant under \( \iota \).

Denote by \( M \) the lattice of rank seventeen, generated by all the \((-2)\) curves in the diagram 4.1. The embedding \( M \hookrightarrow S_X \) is of finite index. A direct computation shows that \( D_M \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \).

**Claim:** The lattice \( S_X \) is two-elementary.

Assuming the claim and letting \( i \) be the index \([S_X : M]\), we get by Proposition II.4 that

\[
|D_M| = 2^5 = i^2 \cdot 2^a \Rightarrow i^2 = 2^{5-a} \Rightarrow a = 1, 3 \text{ or } 5.
\]

If \( a = 5 \), then \( i = 1 \) and \( M = S_X \), which contradicts the assumption that \( S_X \) is two-elementary.

If \( a = 1 \), then by Corollary II.17, \( T_X \cong U \oplus U \oplus \langle -2 \rangle \). Such a \( K3 \) surface is known from Example II.19 to contain finitely many smooth rational curves forming the diagram 2.1 which does not contain a subdiagram similar to 4.1.

We conclude that \( a = 3 \), or equivalently that \( T_X \cong U \oplus U(2) \oplus \langle -2 \rangle \).
Proof of the claim: In order to prove the claim, it is enough to show the existence of an involution \( \vartheta \rightarrow X \) satisfying \( \vartheta^*_S = id_{S_X} \) and \( \vartheta^*_T = -id_{T_X} \).

The cycle

\[
C_{14}' + E_{14}' + C_0 + E_{16} + C_{16}' + E_{26}' + C_{12}' + E_{23}' + t^*C_{12}' + t^*E_{26}' + t^*C_{16}' + t^*E_{16}' + t^*C_0 + t^*E_{14}'
\]

is disjoint from the genus one curve \( E \) (i.e. \( \tau^*C_{15} \)). Consequently it defines the \( I_{14} \) singular fiber of the elliptic fibration, \( f_E : X \overset{\varphi}{\rightarrow} \mathbb{P}^1 \), for which \( S, \nu S, E_{15}' \) and \( t^*E_{15}' \) are sections. Fix \( t^*E_{15}' \) as the zero section. Let \( X \overset{\vartheta}{\rightarrow} X \) be the map induced by the canonical involution on \( X, \) the generic fiber of \( f_E, \) i.e. \( \sigma_{X_n}(x) = -x \) and let \( X \overset{\vartheta_{E_{15}'}}{\rightarrow} X \) be the translation by the section \( E_{15}' \). We can define the involution \( \gamma := \sigma \circ t_{E_{15}'} \), whose action on the transcendental lattice is given by

\[
\gamma^*_T = \sigma^*_T \circ (t_{E_{15}'})^*_T = -id_T \circ id_T = -id_T.
\]

Moreover its action on the Néron-Séveri lattice satisfies \( \gamma^*_S = t^*_S \). Indeed, we observe that \( S + t^*S - E_{15}' \) is linearly equivalent to

\[
E_{14}' + 2(C_0 + E_{16} + C_{16}' + E_{26}' + C_{12}' + E_{23}' + t^*C_{12}' + t^*E_{26}' + t^*C_{16}' + t^*E_{16}' + t^*C_0 + t^*E_{14}' + t^*E_{15}').
\]

The latter belongs to the sublattice of \( S_X \) generated by the section \( t^*E_{15}' \), the class of a fiber of \( f_E \) and the components of the reducible fibers that do not meet \( t^*E_{15}' \). Consequently it is in the kernel of the map \( \beta \) of Theorem IV.2. In other words \( -S + E_{15}' = t^*S \) in \( MW_{f_E}(X) \), or equivalently \( \gamma^*S = t^*S \) and \( \gamma^*E_{15}' = t^*E_{15}' \). It is now clear that \( \gamma^*_M = t^*_M \) and consequently that \( \gamma^*_S = t^*_S \). Define the map \( \theta = \nu \circ \gamma \). By construction \( \theta \) satisfies \( \theta^*_T \circ S_X = -id_T \oplus id_{S_X} \), which proves that \( S_X \) is two-elementary.

**Definition IV.6.** Define the lattice \( E_{8}^{\text{twist}}(-1) \) to be the lattice \( E_8(-1) \) of Example II.2 with the following modification

\[
b(e_i, e_7) = \begin{cases} 
-4 & \text{if } e_i = e_7 \\
2 & \text{if } e_i = e_6 \\
0 & \text{otherwise}
\end{cases}
\]

Note that \( \text{discr} E_{8}^{\text{twist}}(-1) = 4. \)

**Proposition IV.7.** Let \( Y \) be a general Kummer surface and \( N \) be the Nikulin lattice generated by the \( a_i \)'s, then there is a primitive embedding \( E_{8}^{\text{twist}}(-1) \hookrightarrow N \perp \subset S_Y. \)

**Proof.** The elements \( E_{14}, C_{12}, E_{26}, C_{16}, E_{16}, C_0, E_{14}, 2C_{14} + a_5 + a_8 \) generate a sublattice of \( N \perp \) isomorphic to \( E_{8}^{\text{twist}}(-1). \) To prove that the embedding is primitive, it is enough to show that this set can be completed to a basis of \( N \perp \). Let \( T \) be the lattice generated by the elements \( E_{14}, C_{12}, E_{26}, C_{16}, E_{16}, C_0, E_{14}, 2C_{14} + a_5 + a_8 \) and \( 2E_{23} + a_6 + a_7. \) \( T \) is a sublattice of \( N \perp \) of finite index \( i. \) By explicitly taking the determinant of a matrix representing the bilinear form of \( T, \) we find that \( \text{discr} T = 4 \) and therefore, \( 4 = i^2 \text{discr} N \perp. \) Either \( i = 1 \) or \( N \perp \) is unimodular. According to Milnor's classification in Theorem II.6, there is no even unimodular lattice of rank nine. We conclude that \( T = N \perp \) and therefore the embedding is primitive. \( \square \)
**Theorem IV.8.** Let $X$ be a $K3$ surface with $T_X \simeq U(2) \oplus U \oplus (-2)$. Then $X$ has a symplectic involution such that if $X \xrightarrow{T} Y$ is the rational quotient map,

1) $Y$ is a Kummer surface,
2) there is a primitive embedding $E^\text{twist}(1) \hookrightarrow N^\perp$.

In order to prove the theorem, we will first show, using very geometrical arguments, that any $K3$ surface with $T_X \simeq U(2) \oplus U \oplus (-2)$ must contain a diagram of smooth rational curves meeting as in figure 4.1.

Let $\theta$ be the involution on $X$ that satisfies $\sigma_{T_X}^* = -id_{T_X}$ and $\sigma_{S_X}^* = id_{S_X}$. Then it has been shown that $X^\theta := \{ x \in X | \theta(x) = x \} = E \coprod_{i=1}^7 R_i,$ where $E$ is a curve of genus one and the $R_i$’s are disjoint rational curves. The system $|E|$ defines an elliptic fibration $X \xrightarrow{f} \mathbb{P}^1$, with a section and exactly one reducible singular fiber $\mathfrak{F}$ of type $I_{14}$. Moreover $\theta$ acts non-trivially on $\mathbb{P}^1$ under $f$ ([Nik81] p.1424-1430).

The $R_i$ are components of $\mathfrak{F}$. Since $\theta$ preserves all the rational curves, any section of $f$ must meet the singular fiber at a $R_i$ component. Thus the surface $X$ contains the following configuration of $(-2)$-curves

Consider the divisor class

$$F := 2R_2 + 4S_1 + 6R_1 + 3C + 5S_7 + 4R_7 + 3S_6 + 2R_6 + S_5.$$ The system $|F|$ defines another elliptic fibration on $X$, $X \xrightarrow{g} \mathbb{P}^1$. By construction, $g$ has a singular fiber of type $II^*$, a section $R_5$ and a 2-section $S_2$. Moreover, the curves $S_4, R_4, S_3$ and $R_3$ must be part of another reducible fiber of $g$. Denote it by $\mathfrak{F}$. We will show by elimination that $\mathfrak{F}$ must be of type $I_2$.

$\mathfrak{F}$ cannot be of type $I_b$ for $b \geq 4$: Suppose it was, then there would exist a rational smooth curve $R$ meeting $S_4$ at exactly one point, $p$:
Since $\theta^*$ acts as the identity on $S_X$, $\theta(p)$ must also belong to $S_4 \cap R$. But $R$ meets $S_4$ at exactly one point, so $p$ must be fixed by $\theta$. Either $p \in R_5 \cap S_4 \cap R$, which contradicts the fact that $R_5$ is a section of $g$, or $p \in R_4 \cap S_4 \cap R$, which contradicts the assumption that $S_4, R_4, S_3$ and $R_3$ are part of a fiber of type $I_0$.

$\mathcal{J}$ cannot be of type $I_1^*$ : Again, suppose it is, then $S_4$ must be a component of multiplicity one ($S_4$ meets the section $R_5$). Denote by $R$ the other component of multiplicity one of $\mathcal{J}$ that meets $R_4$. Two cases are possible:

1. $R$ does not meet the curve $S_2$:

Then the sixth component of $\mathcal{J}$, $S$ would meet $S_2$ at exactly one point, $q$. Because $q \in S \cap S_2$, its image $\theta(q)$ belongs to $S \cap S_2$. So $\theta(q) = q$ and consequently $q \in R_3 \cap S_2$ or $q \in R_2 \cap S_2$. The first case contradicts the fact that $\mathcal{J}$ is of type $I_1^*$, while in the second case, $S$ would meet a component of another singular fiber, which is absurd.

2. $R$ does meet the curve $S_2$:

Since $S_2$ is a 2-section of $g$ and $S_2 \cdot R_3 = 1$, necessarily $R \cdot S_2 = 1$. Let $p \in R \cap S_2$, then $\theta(p) = p$. It follows that either $p \in R_3 \cap S_2$, which contradicts again the assumption that $\mathcal{J}$ is of type $I_1^*$. Or $p \in R_2 \cap S_2 \cap R$, which implies again that $R$ meets a component of another singular fiber.

$\mathcal{J}$ cannot be of type $I_3^*$ : Suppose it is, then $S_4$ has to be a component of multiplicity one and $R_4, S_3$ and $R_3$ components of multiplicity two. Let $R$ be the fourth component of multiplicity two of $\mathcal{J}$ and $L_1, L_2$ the two components of multiplicity one intersecting it.
Since $L_1$ and $L_2$ do not meet any $R_i$’s nor any $S_i$’s, they must be part of a reducible fiber of the fibration defined by $|E|$. But this fibration is known to have an unique reducible singular fiber.

$\mathcal{J}$ cannot be of type $I_b^*$, with $b \geq 4$: Fibers of type $I_b^*$ have $b + 5$ components. By Theorem IV.2, the number of components of $\mathcal{J}$ cannot exceed 8, so $b \leq 3$.

The same counting argument shows that $\mathcal{J}$ cannot be of type $II^*$.

$\mathcal{J}$ cannot be of type $III^*$: If it was, the group of sections of $g$ would be finite of order $n$, where $n$ would satisfy by Theorem IV.2

$$2^3 = |D_{S_X}| = \frac{1 \cdot 2}{n^2}$$

but no integer satisfies this equation.

Finally $\mathcal{J}$ cannot be of type $IV^*$: Indeed in this case one of the components with multiplicity one will have to be disjoint from every $R_i$ and $S_i$. Such a component must be part of a reducible fiber of the fibration defined by $|E|$. But again, this fibration is known to have an unique reducible singular fiber.

By elimination, we have shown that $\mathcal{J}$ must be of type $I_2^*$. Moreover we have shown that $X$ contains the following configuration of smooth rational curves

By symmetry, the divisor class

$$F' = 2R_7 + 2S_7 + 6R_1 + 3C + 5S_1 + 4R_2 + 3S_2 + 2R_3 + S_3$$

defines an elliptic fibration $X \to \mathbb{P}^1$, for which the curves $R_6 + S_5 + R_5 + S_4$ are part of a reducible fiber of type $I_2^*$

$$\mathcal{S} := S + S_4 + 2R_5 + 2S_5 + 2R_6 + T + T'.$$
Since $R \cdot F' = 0$, the curve $R$ is also part of a reducible fiber clearly disjoint from $\mathfrak{F}$. It follows from Theorem IV.2 that $R$ can only be part of a fiber with two components so generically it is of type $I_2$.

We conclude that the $R_i$’s, the $S_i$’s and the smooth rational curves $R, S, T$ define the diagram

\[
\begin{array}{c}
\begin{array}{c}
\bullet \\
S_1 \\
R_1 \\
S_7 \\
R_7 \\
S_6 \\
S_5 \\
R_6 \\
\end{array}
\begin{array}{c}
\bullet \\
R_2 \\
S_2 \\
R_3 \\
S_3 \\
R_4 \\
S_4 \\
S_5 \\
\end{array}
\begin{array}{c}
\bullet \\
R_5 \\
S \\
\end{array}
\begin{array}{c}
\bullet \\
R \\
\end{array}
\begin{array}{c}
\bullet \\
T
\end{array}
\end{array}
\end{array}
\]

The $I_{10}$ configuration apparent in this diagram defines an elliptic fibration for which $S$ is part of a $I_2$ fiber. By adding the other component of this fiber, we obtain a diagram of smooth rational curves meeting exactly as in Figure 4.1.

We can now prove Theorem IV.8.

\textit{Proof.} In order to construct the symplectic involution $\iota$ we apply the inverse process used in Lemma IV.5 to construct the involution $\theta$. The curves $R, S$ and $T$ are sections of the elliptic fibration $X \xrightarrow{f} \mathbb{P}^1$. Fix $T$ as the zero section. Let $\sigma$ be the map induced by inversion on the generic fiber and $t_R$ be the translation by $R$. The maps $\theta^*$ and $\gamma^*$ commute on $H^2(X, \mathbb{Z})$, we conclude by Theorem II.12 that $\iota$ is an involution. Moreover it follows from the fact that $\iota^*_T = -id_T$ that $\iota$ is a symplectic involution. Its eight fixed points can be easily described. Indeed a point $p$ is fixed by $\iota$ if and only if $\gamma(p) = \theta(p)$. Since $\gamma$ acts trivially on $\mathbb{P}^1$ and $\theta$ acts non trivially, the fixed points of $\iota$ must lie on the two fibers fixed by $\theta$, i.e $E$ or $\mathfrak{F}$. Those points lying on $E$ satisfy $\gamma(p) = \theta(p) = p$ and therefore correspond to the four fixed point of $\gamma_E : E \rightarrow E$. The four remaining fixed points must lie on $\mathfrak{F}$. More precisely, two of them lie on $S_1$ and the other two lie on $R_5$. Indeed, since $S_j \cdot \iota^*(S_j) = 0$ for $j \neq 1$, a point $p$ lying on some $S_j$’s is fixed by $\iota$ if and only if $p \in S_1$ and $\theta(p) = p$. Thus the two fixed points of $\theta_S_1$ are fixed by $\iota$. Similarly, $R_j \cdot \iota^*(R_j) = 0$ for $j \neq 5$ implies that the two fixed points of $\gamma_{R_5}$ are fixed by $\iota$.

Consider $X \xrightarrow{\tau} Y$ the rational quotient map by $\iota$ and denote by $A_1, \ldots, A_8$ the induced even eight. The $A_i$’s and the image of the curves $S_i, R_i, S, R$ and $T$ under
τ form the following diagram of rational curves on $Y$

![Diagram of rational curves on Y]

The surface $Y$ admits therefore an elliptic fibration with a section and whose singular fibers are of the type $6I_2 + I_5^* + I_1$. In the table of [Shi00], we find that the Mordell Weil group of such a fibration must be of order two. It follows now from Proposition IV.4 that $S_Y$ admits a primitive embedding of the Kummer lattice which implies by Theorem II.23 that $Y$ is a Kummer surface. The primitive embedding $E_8^{\text{twist}}(-1) \hookrightarrow N^\perp$ follows from Proposition IV.7.

Let $\Sigma$ be a reducible plane sextic, decomposable as an irreducible quartic $\Gamma$ with an ordinary node and a cusp of type $A_4$ (see [BHPvdV04] p.64), a line $\Lambda_1$ passing through the two singular points of $\Gamma$ and another line $\Lambda_2$ (see figure 4.2).

![Figure 4.2:]

**Theorem IV.9.** The rational quotient map $X \xrightarrow{\tau} Y$ decomposes as

$$
\begin{array}{ccc}
X & \xrightarrow{q_2} & Z \\
\uparrow \tau & & \uparrow q_1 \\
Y & \xrightarrow{q} & \mathbb{P}^2
\end{array}
$$

where $q$ is the canonical resolution of the double cover of $\mathbb{P}^2$ branched along $\Sigma$. The maps $q_1$ and $q_2$ are the canonical resolutions of the double covers branched along $\Gamma$ and $q_1^*(\Lambda_1 + \Lambda_2)$ respectively.

**Proof.** We have shown in the course of proving Theorem IV.8 that the map $\tau$ can be thought as the rational double cover associated to the even eight of Lemma IV.5. Therefore there exists on $X$ a curve of genus two $W$ (i.e. $W = \tau^*C_{13}$) that satisfies

$$W \cap E = \{p_1, p_2, p_3, p_4\}, \quad W \cap R_5 = \{p_5, p_6\} \quad \nu(p_i) = p_i \quad i = 1, \ldots, 6.$$
The quotient $X/\theta$ is a smooth rational surface and applying Hurwitz formula we find that $K^2_{X/\theta} = -7$. After taking the quotient by $\theta$, the image of the nine disjoint curves $S_i$, $i = 1 \cdots 7$, $R$ and $T$ have self intersection $-1$. We may therefore blow them down to a surface $Z$, with $K^2_Z = 2$. The induced map $X \xrightarrow{\eta} Z$ is ramified along the $R_i$'s and $E$ and maps $W$ to a smooth rational curve $C$, with $C^2 = 2$. Since the involution $\iota$ commutes with $\theta$ on $X$, it induces an involution $\iota$ on $Z$. Note that $\iota$ has four fixed points on $C$, $q_2(p_1)$, $q_2(p_2)$, $q_2(p_3)$, $q_2(p_4)$, thus $\iota_C = \text{id}_C$ and the quotient map $Z \to Z/\iota$ is ramified along $C$. Using Hurwitz formula a second time, we find that $K^2_{Z/\iota} = 6$. Moreover under the composition

$$X \xrightarrow{\eta} Z \to Z/\iota$$

the curves $R_2$, $R_3$ and $R_5$ map to rational curves with intersection properties

$$R_5^2 = R_2^2 = -1, \quad R_3^2 = -2, \quad R_5' \cdot R_2' = R_5' \cdot R_3' = 0, \quad R_2' \cdot R_3' = 1.$$

We may successively blow down $R_5'$, $R_2'$ and $R_3'$ to obtain a rational surface whose canonical divisor has self intersection equal to 9, i.e. the projective plane. It is an easy exercise to verify that under the composition

$$X \xrightarrow{\alpha} Z \xrightarrow{\eta_1} \mathbb{P}^2$$

the genus two curve $\psi$ maps to the singular quartic $\Gamma$, the $R_i$'s get contracted to a line $\Lambda_1$ passing through the singularities of $\Gamma$, and the genus one curve $E$ maps to another line $\Lambda_2$. The following proposition completes the proof. \hfill $\square$

**Proposition IV.10.** A surface $Y$ is a general Kummer surface if and only if $Y$ is the canonical resolution of a double cover of $\mathbb{P}^2$ branched along the reducible sextic $\Sigma = \Gamma + \Lambda_1 + \Lambda_2$.

**Proof.** Assume first that $Y$ is a general Kummer surface. Recall that $Y$ admits a double plane model $Y \xrightarrow{\psi} \mathbb{P}^2$ that factors as $Y \xrightarrow{\psi} \tilde{\mathbb{P}}^2 \xrightarrow{\eta} \mathbb{P}^2$ where $\tilde{\mathbb{P}}^2$ is the blowup of $\mathbb{P}^2$ at the fifteen singular points of the union of six lines (see figure 2.2). The image of the curves $a_1, \cdots a_8, C_{15}, E_{15}, C_0, E_{16}, E_{26}, C_{12}, E_{26}, E_{14} C_{14}$ and $C_{13}$ under the map $\psi$ intersect as in the figure 4.3. The number in the parenthesis denotes the self-intersection and $\phi(a_6) = \phi(a_7) = \tilde{C}_{23}$.

We may blow down the curves $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5, \tilde{a}_8, \tilde{E}_{14}, \tilde{E}_{16}, \tilde{E}_{15}, \tilde{E}_{26}, \tilde{E}_{23}, \tilde{C}_{14}, \tilde{C}_{23}, \tilde{C}_{12}$ and $\tilde{C}_{16}$. This sequence of blow-downs defines a Cremona transformation

$$\begin{array}{ccc}
\mathbb{P}^2 & \xrightarrow{\epsilon} & \tilde{\mathbb{P}}^2 \\
\text{-----} & \xi & \text{-----} \\
\mathbb{P}^2 & \xrightarrow{\eta} & \mathbb{P}^2
\end{array}$$

which maps the six lines of $S$ as follows

$$\xi(l_0) = \Lambda_1 \quad \xi(l_3) = \Gamma \quad \xi(l_5) = \Lambda_2$$
and the lines $l_2$, $l_4$ and $l_6$ get contracted to the singular points of $\Gamma$.

Conversely, let $Y_0 \xrightarrow{\eta_1} \mathbb{P}^2$ be the blowup of the $D_9$ singularity of $\Sigma$ (line passing through a cusp of type $A_4$) and denote by $\Upsilon_1$ the exceptional divisor. By construction ([BHPVdV04] p.87), the canonical resolution of the double cover defined by $\Sigma$ is the canonical resolution of the double cover defined by

$$\Sigma_1 = \eta_1^* \Sigma - 2 \Upsilon_1 = \Gamma + 2 \Upsilon_1 + \Lambda_1 + \Upsilon_1 + \Lambda_2 - 2 \Upsilon_1 = \Gamma + \Lambda_1 + \Lambda_2 + \Upsilon_1$$

where we use the same symbol to denote a curve and its proper transform. Similarly, if $Y_1 \xrightarrow{\eta_2} Y_0$ is the blowup of the point $\Gamma \cap \Upsilon_1$, then the canonical resolution of the double cover defined by $\Sigma_1$ is the canonical resolution of the double cover defined by

$$\Sigma_2 = \eta_2^* \Sigma_1 - 2 \Upsilon_2 = \Gamma + \Lambda_1 + \Lambda_2 + \Upsilon_1 + \Upsilon_2.$$

Blow up the simple node of $\Gamma$ and denote the exceptional divisor by $\Upsilon_3$. We obtain the branch divisor

$$\Sigma_3 = \Gamma + \Lambda_1 + \Lambda_2 + \Upsilon_1 + \Upsilon_2 + \Upsilon_3.$$

Since the singularities of $\Sigma_3$ are now all of type $A_1$, the exceptional divisors coming from their resolution will not be part of the branch locus along which we will take the double cover. The resolution of $\Sigma_3$ gives a graph identical to the one of 4.3 in $\mathbb{P}^2$ where $\Gamma + \Lambda_1 + \Lambda_2 + \Upsilon_1 + \Upsilon_2 + \Upsilon_3$ corresponds exactly to $\tilde{C}_{13} + \tilde{C}_0 + \tilde{C}_{15} + \tilde{C}_{16} + \tilde{C}_{12} + \tilde{C}_{14}$. \hfill \square

### 4.3 K3 Surfaces with $T_X \simeq U(2) \oplus U(2) \oplus \langle -2 \rangle$

This section is very similar to the previous one and uses the same notation. Indeed we give another new even eight on a general Kummer surface and show that this time the corresponding rational double cover $X$ has its transcendental lattice isomorphic to $U(2) \oplus U(2) \oplus \langle -2 \rangle$. Again we exhibit an elliptic fibration on $X$ such that the
symplectic involution is obtained as the composition of the natural involution on the general fiber with an involution acting non-trivially on the base.

**Proposition IV.11.** Let $Y$ be a general Kummer surface. Then there exists an elliptic fibration on $Y$ whose singular fibers are of the type $4I_2 + I^*_2 + I^*_1 + I_1$ and whose Mordell-Weil group is cyclic of order two.

**Proof.** Consider the divisor class

$$D = 3(L - E_0) - 2E_{12} - (E_{13} + E_{24} + E_{45} + E_{46} + E_{56})$$

Geometrically, $D$ can be represented by the proper inverse image, under the map $Y \overset{\phi}{\to} \mathbb{P}^2$, of a cubic passing through $p_{13}, p_{24}, p_{45}, p_{46}, p_{56}$ having a double point at $p_{12}$. By counting parameters, we immediately see that such a cubic curve exists. Generically its proper transform $D$ is smooth and irreducible and satisfies $D^2 = 0$.

Let $Y \overset{f}{\to} \mathbb{P}^1$ be the elliptic fibration defined by the linear system $|D|$. The divisors

1. $F_1 = (L - E_0) - (E_{12} + E_{45}) + 2(L - E_0) - (E_{12} + E_{13} + E_{24} + E_{46} + E_{56})$

2. $F_2 = 2(L - E_0) - (E_{12} + E_{13} + E_{24} + E_{45} + E_{56}) + e_1$

3. $F_3 = 3(L - E_0) - 2E_{12} - (E_{13} + E_{24} + E_{36} + E_{45} + E_{46} + E_{56}) + E_{36}$

4. $F_4 = E_{35} + 3(L - E_0) - 2E_{12} - (E_{13} + E_{24} + E_{45} + E_{56} + E_{46} + E_{35})$

define four fibers of type $I_2$.

The divisor

$$F_5 = (L - E_0) - (E_{12} + E_{56}) + E_{14} + 2(C_0 + E_{14} + C_{14}) + E_{15} + E_{16}$$

defines a fiber of type $I^*_2$.

The divisor

$$F_6 = e_6 + e_7 + 2(C_{12} + E_{23}) + E_{26} + E_{25}$$

defines a fiber of type $I^*_1$. Since $C_{15} \cdot D = C_{16} \cdot D = 1$, the rational curves $C_{15}$ and $C_{16}$ are sections of $f$.

$$\sum_{i=1}^{6} \chi_{top}(F_i) = 23 \Rightarrow \text{there is one more singular fiber } F_7 \text{ of type } I_1.$$

Finally, by Theorem IV.2, we find that $MW_f(X) \simeq \mathbb{Z}/2$. 


Set \( b_6 = e_6 \quad b_7 = e_7 \quad b_8 = E_{34} \).

**Lemma IV.12.** The \( b_i \)'s form an even eight on \( Y \). If \( X \xrightarrow{\tau} Y \) is the associated rational map of degree two, then \( T_X \simeq U(2) \oplus U(2) \oplus \langle -2 \rangle \).

**Proof.** A direct computation shows that
\[
\sum_{i=1}^{8} b_i = 2C_{13} + 2E_{35} + 2E_{34} + 2\{4(L - E_0) - (3E_{12} + 2E_{45} + 2E_{36} + E_{13} + E_{24} + E_{46})\} \in 2S_Y
\]

We will apply the same strategy as in Lemma IV.5 and analyze the fibration \( X \xrightarrow{g} \mathbb{P}^1 \) defined by \( \tau^*D \). Let \( X \xrightarrow{i} X \) be the covering symplectic involution. Likewise in Lemma IV.5, \( \tau^*F_i, i = 1, 2, 3, 4 \) are nodal curves preserved by \( i \) which implies that \( i \) acts trivially on \( \mathbb{P}^1 \) under \( g \) and that \( \tau^*D \) cannot split. As for the remaining singular fibers

\( \tau^*F_5 \) becomes of type \( I_4^* \), \( \tau^*F_6 \) becomes of type \( I_2^* \), \( \tau^*F_7 \) becomes of type \( I_2 \).

Finally \( \tau^*C_{15} \) and \( \tau^*C_{16} \) become 2-sections of \( g \). We conclude that there exists an embedding of lattices of finite cokernel
\[
U(2) \oplus D_8 \oplus D_6 \oplus A_1 \hookrightarrow S_X, \quad \text{and } D_{U(2) \oplus D_6 \oplus A_1} \simeq \langle \mathbb{Z}/2 \rangle^{\oplus 7}.
\]

The overlattice \( S_X \) must be a two-elementary lattice. Note also that \( g \) does not have a section. Indeed, if \( C \cdot \tau^*D = 1 \), then \( i^*C \) also satisfies
\[
i^*C \cdot i^*(\tau^*D) = i^*C \cdot \tau^*D = 1.
\]
Let \( i^*C + C = \tau^*\tilde{C} \) for some \( \tilde{C} \in S_Y \). Then
\[
2\tilde{C} \cdot D = \tau^*\tilde{C} \cdot \tau^*D = 2 \Rightarrow \tilde{C} \cdot D = 1.
\]
So either \( C + i^*C = \tau^*C_{15} \) or \( C + i^*C = \tau^*C_{16} \). But we have seen that \( C_{15} \) and \( C_{16} \) do not split as they each meet the \( b_i \)'s at two points. This shows indeed that \( g \) does not have a section.

Let \( J(X) \) be the Jacobian fibration of \( g \) (see Definition IV.16 below). We will see in Lemma IV.17 of the next section that there is an embedding
\[
T_X \hookrightarrow T_{J(X)}, \quad \text{with } T_{J(X)}/T_X \simeq \mathbb{Z}/2.
\]
Let \( a \) be the length of \( D_{S_X} \), i.e. \( D_{S_X} \simeq \langle \mathbb{Z}/2 \rangle^a \).

If \( a = 1 \), then by Corollary II.17, \( T_X \simeq U \oplus U \oplus \langle -2 \rangle \) and \( X \) contains finitely many rational curves meeting as in Figure 2.1 which do not define any \( I_4^* + I_2^* + I_2 \) configuration.

If \( a = 3 \), then \( T_{J(X)} \) has to be isomorphic to \( U \oplus U \oplus \langle -2 \rangle \) and we get the same contradiction as in the previous case applied to the surface \( J(X) \). We conclude that
\[
T_X \simeq U(2) \oplus U(2) \oplus \langle -2 \rangle.
\]
\[\square\]
Theorem IV.13. Let \( X \) be a K3 surface with \( T_X \cong U(2) \oplus U(2) \oplus \langle -2 \rangle \). Then \( X \) has a symplectic involution such that if \( X \xrightarrow{\tau} Y \) is the rational quotient map, then \( Y \) is a general Kummer surface.

Proof. From a purely lattice theoretical point of view, we deduce from Lemma IV.12 that there exists an embedding

\[
U(2) \oplus D_8 \oplus D_6 \oplus A_1 \hookrightarrow S_X.
\]

Thus \( X \) admits an elliptic fibration \( X \xrightarrow{g} \mathbb{P}^1 \) with singular fibers of type \( I_4^* + I_2^* + I_2 \), i.e. it contains the following diagram of smooth rational curves

\[
\begin{array}{c}
S_1 \\
\downarrow \\
R_1 \quad S_2 \quad R_2 \quad S_3 \\
\downarrow \\
R_3 \quad S_4 \quad R_4 \\
\downarrow \\
S_5 \quad S_6 \quad S_7 \\
\downarrow \\
R_5 \quad S_8 \quad R_6 \\
\downarrow \\
S_9 \quad S_{10} \quad S_{11} \\
\end{array}
\]

We shall see that the multisection index (see Definition II.4) of \( g \) is two. Note that by construction \( g \) admits a 2-section that we call \( R_6 \). Without loss of generality let \( R_6 \cdot S_2 = R_6 \cdot S_6 = R_6 \cdot S_8 = R_6 \cdot S_{11} = 1 \).

Let \( X \xrightarrow{\theta} X \) be the involution acting trivially on \( S_X \).

Nikulin showed in [Nik81] that \( X^\theta = \{ x \in X | \theta(x) = x \} \) consists of the union of seven disjoint smooth rational curves. In fact, the curves \( R_1, R_2, R_3, R_4, R_5, R_6 \) belong to \( X^\theta \). Indeed

\[
R_1 \cdot S_1 = R_1 \cdot S_2 = R_1 \cdot S_3 = 1
\]

imply that there exist three points on \( R_1 \) fixed by \( \theta \), consequently \( R_1 \subset X^\theta \). The same argument applies for \( R_3, R_4, R_5, R_6 \).

Assume that \( R_2 \not\subset X^\theta \) and denote by \( p \) the point of \( R_2 \cap S_3 \), because \( R_2 \cdot S_3 = 1 \), \( \theta(p) = p \), so there exists \( C \subset X^\theta \) such that

\[
p \in C \cap R_2 \cap S_3.
\]

If \( F \) denotes the class of a fiber of \( g \), then \( C \cdot F \geq 4 \). The curves \( S \) and \( S' \) cannot be fixed by \( \theta \) as they don’t meet \( S_1, S_5, S_7 \) and \( S_{10} \) which must intersect \( X^\theta \). It implies that \( C \cdot (S + S') \leq 4 \), so \( C \cdot (S + S') = 4 \). Similarly, if

\[
q = R_2 \cap S_4,
\]

then there exists \( C' \subset X^\theta \), such that \( C' \cdot F \geq 4 \).

If \( C \neq C' \), then \( C' \cdot (S + S') = 0 \) which is absurd. If \( C = C' \), then \( C \cdot F \geq 8 \), which contradicts \( C \cdot (S + S') = 4 \). So \( R_2 \subset X^\theta \).

Denote by \( R_7 \) the remaining curve of \( X^\theta \). Necessarily

\[
(R_6 + R_7) \cdot S = (R_6 + R_7) \cdot S' = 2
\]
and $R_7$ is also a 2-section satisfying

$$R_7 \cdot S_1 = R_7 \cdot S_5 = R_7 \cdot S_7 = R_7 \cdot S_{10} = 1.$$ 

Observe now that

$$D_1 = 2R_6 + S_2 + S_6 + S_8 + S_{11} \overset{\text{lin}}{\sim} 2R_7 + S_1 + S_5 + S_7 + S_{10} = D_2.$$ 

The divisors $D_1$ and $D_2$ are two $I_0^*$ fibers of an elliptic fibration $X \rightarrow \mathbb{P}^1$. The curves $S_3, R_2, S_4$ and $S_9$ are components of the other singular fibers and $R_1, R_3, R_4, R_5$ are sections of $h$.

Fix $R_4$ as the zero section. Let $\sigma$ be the map induced by inversion of the generic fiber and $t_{R_5}$ be the translation by $R_5$. The map $\gamma = \sigma \circ t_5$ is an involution on $X$ such that $\gamma^*_X = -id_X$. Set $\iota := \theta \circ \gamma$. As the maps $\theta^*$ and $\gamma^*$ commute on $H^2(X, \mathbb{Z})$ and $\gamma^* \circ \theta^*_X = id_X$ we conclude by Theorem II.12 that $\iota$ is a symplectic involution. Note that

$$R_1 + S_3 + R_2 + S_4 + R_3 + S_3 + R_7 + S_1 \overset{\text{lin}}{\sim} R_4 + S_9 + R_5 + S_{11} + R_6 + S_8$$

as they both define two disjoint $I_8$ and $I_6$ fibers of some other fibration. Therefore $-R_1 + R_5 = R_3$ in $MW_h(X)$ and consequently $\iota^* R_1 = \gamma^* R_1 = R_3$. Clearly $\iota^* R_4 = \gamma^* R_4 = R_5$.

The involution $\gamma^*$ acts trivially on the curves $R_6, R_7, R_2$ and $S_9$. Consequently so does $\iota$ and each of those curves contains two of the eight fixed points of $\iota$. Finally, observe that

$$\iota^* S_1 = S_5, \quad \iota^* S_2 = S_6, \quad \iota^* S_3 = S_4, \quad \text{and} \quad \iota^* S_7 = S_{10} \quad \iota^* S_8 = \iota^* S_{11}.$$ 

Consider $X \overset{\tau}{\rightarrow} Y$ the rational quotient map by $\iota$ and denote by $B_1, \ldots B_8$ the induced even eight. Then $Y$ contains the following diagram of $-2$-curves

\begin{center}
\begin{tikzpicture}
  \node (B1) at (0,0) [circle,fill,inner sep=2pt] {B_1};
  \node (B2) at (1,0) [circle,fill,inner sep=2pt] {B_2};
  \node (B3) at (1,-1) [circle,fill,inner sep=2pt] {B_3};
  \node (B4) at (0,-1) [circle,fill,inner sep=2pt] {B_4};
  \node (B5) at (-1,0) [circle,fill,inner sep=2pt] {B_5};
  \node (B6) at (-1,-1) [circle,fill,inner sep=2pt] {B_6};
  \node (B7) at (2,0) [circle,fill,inner sep=2pt] {B_7};
  \node (B8) at (2,-1) [circle,fill,inner sep=2pt] {B_8};
  \draw (B1) -- (B2);
  \draw (B2) -- (B3);
  \draw (B3) -- (B4);
  \draw (B4) -- (B5);
  \draw (B5) -- (B1);
  \draw (B1) -- (B7);
  \draw (B7) -- (B8);
  \draw (B8) -- (B2);
  \draw (B2) -- (B6);
  \draw (B6) -- (B3);
  \draw (B3) -- (B4);
  \draw (B4) -- (B5);
\end{tikzpicture}
\end{center}

By inspection of the diagram, we see that $Y$ admits an elliptic fibration with a section and whose singular fibers are of the type $4I_2 + I_0^* + I_0^*$. In the table of [Shi00], we find that the Mordell Weil group of such a fibration must be of order two. It follows now from Theorem IV.9 that $S_Y$ admits a primitive embedding of the Kummer lattice which implies by Theorem II.23 that $Y$ is a Kummer surface. \qed
Let $\mathcal{X}$ be a reducible plane sextic, decomposable as three lines $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ meeting at a point, a line $\mathcal{L}_4$ and a conic $\mathcal{W}$ tangent to $\mathcal{L}_4$ (see figure 4.4).

**Proposition IV.14.** A surface $Y$ is a general Kummer surface if and only if $Y$ is the canonical resolution of a double cover of $\mathbb{P}^2$ branched along the reducible sextic $\mathcal{X}$. Moreover the double cover $Y \xrightarrow{\pi} \mathbb{P}^2$ contracts the $b_i$'s of Lemma IV.12.

**Proof.** We proceed exactly as in Proposition IV.10. Consider the images of the curves $b_1, \ldots, b_8, C_{15}, E_{15}, C_0, E_{14}, C_{14}, C_{13}, E_{25}, E_{16}, E_{23}, C_{12}, E_{26}, C_{16}, C_{14}, E_{25}, E_{16}, C_{12}, C_0$. Since $\tilde{\mathbb{P}}^2$ is the blow up of $\mathbb{P}^2$ at fifteen points, the resulting surface is, the blow up of $\mathbb{P}^2$ at a point, i.e $\mathbb{F}_1$. Recall that

$$\text{Pic}\mathbb{F}_1 = \mathbb{Z}H \oplus \mathbb{Z}F, \quad H^2 = 1, \quad H \cdot F = 1, \quad F^2 = 0.$$ 

Under this sequence of blowdowns $\tilde{C}_{15}$ and $\tilde{C}_{16}$ become linearly equivalent to $H$ while $\tilde{C}_{14}$ and $\tilde{C}_{12}$ become linearly equivalent to $F$. Therefore the divisor $\tilde{C}_{14} - \tilde{C}_{15} \sim E_p$, where is $E_p$ is the exceptional divisor of $\mathbb{F}_1$ (we use the same symbol to denote the curves $\tilde{C}_{13}, \tilde{C}_{14}, \tilde{C}_{15}, \tilde{C}_{16}$ and their images in $\mathbb{F}_1$.) By keeping track of the intersections all along the blow-downs, we see that, on $\mathbb{F}_1$, $\tilde{C}_{13} \cdot \tilde{C}_{14} = \tilde{C}_{13} \cdot \tilde{C}_{15} = 2$. It follows that $\tilde{C}_{13} \cdot E_p = 0$. Consider the contracting morphism $\mathbb{F}_1 \to \mathbb{P}^2$, then $\tilde{C}_{13}$ becomes a conic $\mathcal{W}$. The curves $\tilde{C}_{14}, \tilde{C}_{15}, \tilde{C}_{16}$ become three lines, $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$, meeting at a point and the curve $\tilde{C}_{12}$ becomes a line $\mathcal{L}_4$ tangent to $\mathcal{W}$. Again we have constructed a Cremona transformation

$$\begin{align*}
\tilde{\mathbb{P}}^2 & \xrightarrow{\epsilon} \mathbb{P}^2 \xrightarrow{\eta} \tilde{\mathbb{P}}^2 \\
\mathbb{P}^2 & \xrightarrow{- - - - - - - - -} \mathbb{P}^2
\end{align*}$$

which maps the six lines of $S$ as follows

$$\xi(l_3) = \mathcal{W}, \quad \xi(l_2) = \mathcal{L}_4, \quad \xi(l_1) = \mathcal{L}_1, \quad \xi(l_5) = \mathcal{L}_2, \quad \xi(l_6) = \mathcal{L}_3.$$

and the line $l_0$ gets contracted to the point of intersection $\mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_3$. 

![Figure 4.4:](image-url)
Conversely, let $Y_0 \to \mathbb{P}^2$ be the blowup of the point $p \in \mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_3$ and denote by $\Upsilon_1$ the exceptional divisor. By construction ([BHPVdV04] p.87), the canonical resolution of the double cover defined by $X$ is the canonical resolution of the double cover defined by

$$X_1 = \eta^* X - 2Y_1 = W + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \Upsilon_1$$

(again we use the same symbol to denote a curve and its proper transform). Since the singularities of $X_1$ are now all of type $A_1$, the exceptional divisors coming from their resolution will not be part of the branch locus along which we will take the double cover. Moreover, the canonical resolution of $X_1$ gives rise to a graph of rational curves in $\tilde{\mathbb{P}}^2$ meeting exactly as the curves $\tilde{b}_1, \cdots, \tilde{b}_8, \tilde{E}_{14}, \tilde{E}_{15}, \tilde{E}_{23}, \tilde{E}_{26}, \tilde{E}_{25}, \tilde{E}_{16}, \tilde{C}_{12}, \tilde{C}_{0}, \tilde{C}_{13}, \tilde{C}_{14}, \tilde{C}_{15}, \tilde{C}_{16}$. The divisor $W + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \Upsilon_1$ corresponds to $\tilde{C}_{13} + \tilde{C}_{14} + \tilde{C}_{15} + \tilde{C}_{16} + \tilde{C}_{12} + \tilde{C}_0$. It is clear now that the double cover along such a divisor is a general Kummer surface.

**Lemma IV.15.** The map $X \overset{\tau}{\longrightarrow} Y$ decomposes as

$$\begin{array}{ccc}
X & \overset{\pi_2}{\longrightarrow} & \mathbb{P}^1 \times \mathbb{P}^1 \\
\downarrow \tau & & \downarrow \pi_1 \\
Y & \overset{\pi}{\longrightarrow} & \mathbb{P}^2
\end{array}$$

where $\pi$ is the canonical resolution of the double cover of $\mathbb{P}^2$ branched along $X$. The maps $\pi_1$ and $\pi_2$ are the canonical resolutions of the double covers branched along $W$ and $\pi_1^*(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4)$ respectively.

**Proof.** Let $\mathbb{P}^1 \times \mathbb{P}^1 \overset{\pi_1}{\longrightarrow} \mathbb{P}^2$ be the double cover defined by $W$. Note that the line $\mathcal{L}_4$ splits under the cover and that $\pi_1^*(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4)$ is a divisor class of bidegree $(4, 4)$. The canonical resolution of the double cover defined by this even divisor is a $K3$ surface which admits a rational map $X \overset{\tilde{\tau}}{\longrightarrow} Y$. It is an easy exercise to check that the corresponding even eight corresponds to the $b_i$’s of Lemma IV.12 and therefore $\tilde{\tau} = \tau$. 

### 4.4 Torsors and the Jacobian Fibration

**Definition IV.16.** [CD89]

Let $X$ be an elliptic $K3$ surface $X \overset{f}{\to} \mathbb{P}^1$ of multisection index $l$. There exists a unique elliptic $K3$ surface $J(X) \overset{g}{\to} \mathbb{P}^1$ with the following properties:

- $J(X) \overset{g}{\to} \mathbb{P}^1$ has a section.
- Each fiber of $f$ is isomorphic to a fiber of $g$.

The generic fiber $X_\eta$ of $f$ is then a principal homogeneous space with respect to the generic fiber $J(X)_\eta$ of $g$, i.e. $X_\eta$ is a torsor over $J(X)_\eta$. The surface $J(X)$ is called the Jacobian fibration of $X \overset{f}{\to} \mathbb{P}^1$. 

Theorem IV.17. [Keu00] Let $f : X \rightarrow \mathbb{P}^1$ be an elliptic K3 surface with multisection index $l$. Let $F$ be the divisor class of a fiber. Then $T_X$ embeds in $T_J(X)$ where the cokernel is cyclic of order $l$.

Consider the diagram

$$
\begin{array}{c}
Z \\
\mu \downarrow \downarrow \tau \downarrow \\
X \\
Y
\end{array}
$$

where $Y$ is a general Kummer surface and $\mu$ is the rational double cover defined by the $e_i$’s in chapter 1 and $\tau$ is the rational double cover defined by the $a_i$’s in Lemma IV.5.

Theorem IV.18. There exists an elliptic fibration $Y \rightarrow [D] \rightarrow \mathbb{P}^1$ such that $\mu^*D$ and $\tau^*D$ induce two elliptic fibrations.

where $\tau^*D$ has multisection index 2 and $Z \rightarrow [\mu^*D] \rightarrow \mathbb{P}^1$ is its Jacobian fibration.

Proof. Recall the divisor

$$D = 5(L - E_0) - 3E_{12} - 2(E_{13} + E_{46} + E_{56}) - (E_{24} + E_{25} + E_{36} + E_{45}) \in S_Y$$

of Proposition IV.4. We have shown in the proof of Lemma IV.5 that if $X \rightarrow Y$ is the rational double cover associated to the $a_i$’s, then the linear system $\tau^*D$ defines an elliptic fibration $X \rightarrow \mathbb{P}^1$ with singular fibers of type $I_{10}^* + I_{13}$. We will show that the multisection index of $g$ is two. If $C$ is a section of $g$ then

$$\iota^*C \cdot \iota^*\tau^*D = \iota^*C \cdot \tau^*D = 1.$$  

Since $C \neq \iota^*C$ ($\iota^*$ acts trivially on the base of the fibration), $\iota^*C + C = \tau^*\tilde{C}$ for some $\tilde{C} \in S_Y$. Then

$$2\tilde{C} \cdot D = \tau^*\tilde{C} \cdot \tau^*D = 2 \Rightarrow \tilde{C} \cdot D = 1.$$  

So either $C + \iota^*C = \tau^*C_{14}$ or $C + \iota^*C = \tau^*C_{15}$. But we have seen that $C_{14}$ and $C_{15}$ do not split as they each meet the $a_i$’s at two and four points respectively. By the fact that $\tau^*D \cdot C_{14} = 2$, we conclude that the multisection index of $g$ is two. Let $J(X)$ be the Jacobian fibration of $g$. According to Theorems IV.17, II.13 and III.9, there exist Hodge embeddings of index two $T_X \hookrightarrow T_{J(X)}$ and $T_X \hookrightarrow T_Z$. The lattice $T_X$ admits only one overlattice up to isomorphism, so $T_{J(X)} \simeq T_Z$ and by Theorem II.13 $J(X) \simeq Z$. Hence $J(X)$ has a Shioda-Inose structure. In Theorem III.13, Naruki describes the even eight corresponding to the rational map $J(X) \rightarrow Y$. It is then easy to see that $\mu^*D$ is an elliptic fibration with a section (in fact two sections) whose singular fibers are of the type $I_{10}^* + I_{13}$. \qed
Remark IV.19. Denote by \(\alpha\) and \(\alpha'\) the \(\mathbb{Q}\)-divisors \(\frac{D-2e_5}{2}\) and \(\frac{D-2e_8}{2}\) respectively. The composition of the maps \(\omega_\alpha : S_Y \rightarrow S_Y \otimes \mathbb{Q}\) and \(\omega_{\alpha'} : S_Y \rightarrow S_Y \otimes \mathbb{Q}\) defined by \(\omega_\alpha(x) = x + (x \cdot \alpha)\alpha\) and \(\omega_{\alpha'}(x) = x + (x \cdot \alpha')\alpha'\) respectively sends the \(e_i\)'s to the \(a_i\)'s. However \(\omega_\alpha\) and \(\omega_{\alpha'}\) do not belong to \(O(S_Y)\) as they do not define a reflection with respect to an element in \(S_Y\).

It is in fact possible to reproduce this construction for the rational double cover defined by the \(b_i\)'s in Lemma IV.12. Indeed recall that if \(D\) denotes now the divisor of Proposition IV.11

\[
D = 3(L - E_0) - 2E_{12} - (E_{13} + E_{24} + E_{45} + E_{46} + E_{56}) \in S_Y,
\]

and if \(X \xrightarrow{\tau} Y\) denotes the rational double cover defined by the \(b_i\), then \(\tau^*D\) defines an elliptic fibration of multisection index two and singular fibers of type \(I_4 + I_2 + I_2\). It is again possible to construct a composition of reflections of \(S_Y\) with respect to \(\mathbb{Q}\)-divisors, which maps the \(b_i\)'s to another even eight such that if \(Z \xrightarrow{\mu} Y\) is the corresponding double cover, then \(\mu^*D\) defines an elliptic fibration on \(Z\) with a section (two sections in fact) and with singular fibers of type \(I_4 + I_2 + I_2\). It is highly expected that this \(Z\) is the Jacobian fibration of \(X \xrightarrow{\tau^*D} \mathbb{P}^1\).
CHAPTER V

Isogeny of Abelian Surfaces

5.1 Isogeny of Abelian Surfaces

Definition V.1. An isogeny of abelian varieties is a surjective homomorphism $\mathbb{C}^n/\Lambda' \to \mathbb{C}^n/\Lambda$ with finite kernel.

Remark V.2. In this section, a lattice $\Lambda$ will have a slightly different meaning. It will denote a free $\mathbb{Z}$-module of rank $2n$, embedded into $\mathbb{C}^n$, such that $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{C}^n$. A morphism of lattices, $\Lambda' \to \Lambda$ is induced by a $\mathbb{C}$-linear map $\mathbb{C}^n \to \mathbb{C}^n$.

Let $\mathbb{C}^2/\Lambda' \to \mathbb{C}^2/\Lambda$ be an isogeny of abelian surfaces of degree two. The induced map $\mathbb{C}^2 \to \mathbb{C}^2$ on the universal cover defines an embedding of lattices $\Lambda' \to \Lambda$ of index two. On the other hand, embeddings of lattices $\Lambda' \to \Lambda$ of index two are in one to one correspondence with the hyperplanes of the $\mathbb{F}_2$-vector space formed by the two-torsion points of $\mathbb{C}^2/\Lambda$. Indeed a hyperplane in $\Lambda/2\Lambda$ is of the form $\Lambda'/2\Lambda$, where $\Lambda' \subset \Lambda$ is a sublattice of index two.

5.2 Kummer Surfaces Covering a Kummer Surface

Recall from Theorem II.22 that any $K3$ surface $Y$ containing sixteen disjoint smooth rational curves is a Kummer surface. The proof of this theorem relies on the following deep fact:

Denote by $C_1 \ldots C_{16}$ sixteen disjoint smooth rational curves on $Y$, by $I = \{1, \ldots, 16\}$ the set of indices for the curves $C_i$’s and by $Q = \{ M \subset I \mid \frac{1}{2} \sum_{i \in M} C_i \in S_Y \}$, then for every $M$ in $Q$, we have $\#|M| = 8$ or $16$ and there exists on $I$ a unique 4-dimensional affine geometry structure over $\mathbb{F}_2$, whose hyperplanes consist of the subsets $M \in Q$ containing eight elements.

Indeed, the existence of such a 4-dimensional affine geometry implies that $I \in Q$ or equivalently that $\sum_{i=1}^{16} C_i \in 2S_Y$. The corresponding rational double cover $A \to Y$ realizes $Y$ as $\text{Kum}(A)$.
Remark V.3. By uniqueness, the affine geometry on $I$ corresponds to the one existing on $A_2$, the set of 2-torsion points on $A$.

As a corollary to this result, we obtain a lower bound for the number of even eight’s on a Kummer surface.

**Corollary V.4.** Let $Y$ be a Kummer surface, then there exist at least thirty even eight’s on $Y$.

**Proof.** Let $C_1, \ldots, C_{16}$ be sixteen disjoint smooth rational curves on $Y$ realizing $Y$ as $\text{Kum}(A)$. By Nikulin’s result, the subsets

$$\{C_{i_1}, \cdots, C_{i_8}\} \subset \{C_1 \cdots C_{16}\}$$

such that $\sum_{j=1}^{8} C_{ij} \in 2S_Y$

correspond to the affine hyperplanes of $A_2$. Since $A_2$ is a 4-dimensional vector space over $\mathbb{F}_2$, there exist thirty affine hyperplanes and consequently thirty such even eight’s.

**Theorem V.5.** Let $Y$ be a Kummer surface and let $C_1, \ldots, C_8$ be an even eight on $Y$. The rational double cover of $Y$ branched over $\sum_{i=1}^{8} C_i$, $X$, is a Kummer surface if there exist eight disjoint rational curves on $Y$, $C_1', \ldots, C_8'$, satisfying

$$C_i \cdot C_j' = 0 \text{ for all } i, j.$$

Moreover, by taking the rational double cover branched along $\sum_{i=1}^{8} C_i$, we obtain the following commutative diagram:

\[
\begin{array}{ccc}
B & \xrightarrow{q} & A \\
\downarrow \pi_B & & \downarrow \pi_A \\
X = \text{Kum}(B) & \rightarrow & \text{Kum}(A) = Y
\end{array}
\]

where $q$ is an isogeny of degree two.

**Proof.** The curves $C_1, \ldots, C_8, C_1', \ldots, C_8'$ are sixteen disjoint smooth rational curves on $Y$. By Nikulin’s result, they define an abelian surface $A$ and a rational map of degree two, $\pi_A : A \dashrightarrow Y \simeq \text{Kum}(A)$. Since the curves $C_1' \ldots C_8'$ form an even eight, they correspond to an affine hyperplane, $H'$, in $A_2$. Up to translation we can fix the origin on $A$ so that it is contained in $H'$. Let $\Lambda'$ be the sublattice corresponding to $H'$ and let $\{h_1, h_2, h_3, v\}$ be a basis for $\Lambda'$ such that $h_1, h_2, h_3, v \in \Lambda/2\Lambda$. The canonical inclusion $\Lambda' \hookrightarrow \Lambda$, induces the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{C}^2/\Lambda' & \xrightarrow{q} & \mathbb{C}^2/\Lambda \\
\downarrow \pi_{B'} & & \downarrow \pi_A \\
\text{Kum}(\mathbb{C}^2/\Lambda') & \rightarrow & \text{Kum}(\mathbb{C}^2/\Lambda)
\end{array}
\]
where $q$ is an isogeny of degree two. The covering involution of $q$ is given by the translation by the 2-torsion point $[v]$:

$$\tau : C^2/\Lambda' \to C^2/\Lambda', \quad \tau([z]) = [z] + [v]$$

It induces the symplectic involution on $\text{Kum}(C^2/\Lambda')$:

$$\sigma : \text{Kum}(C^2/\Lambda') \to \text{Kum}(C^2/\Lambda').$$

The map $\sigma$ has exactly eight fixed points, namely the projection of the sixteen points on $C^2/\Lambda'$ satisfying

$$[z] + [v] = -[z], \text{ or equivalently } 2[z] = [v].$$

If $M = \{[z] \in C^2/\Lambda' | 2[z] = [v]\}$, then $q(M) = A_2 - H'$. Indeed if $[z] \in M$, then

$$2z - v = n_1 h_1 + n_2 h_2 + n_3 h_3 + n_4 2v \text{ with } n_i \in \mathbb{Z}.$$  

It implies that

$$z - \frac{v}{2} = n_1 \frac{h_1}{2} + n_2 \frac{h_2}{2} + n_3 \frac{h_3}{2} + n_4 v \text{ with } n_i \in \mathbb{Z}.$$  

Consequently

$$q([z]) = \frac{v}{2} + \epsilon_1 \frac{h_1}{2} + \epsilon_2 \frac{h_1}{2} + \epsilon_3 \frac{h_1}{2} \text{ with } \epsilon_i = 0 \text{ or } 1 \Rightarrow q([z]) \in A_2 - H'.$$

In other words, the affine hyperplane $A_2 - H'$ corresponds in $\text{Kum}(C^2/\Lambda)$ to the even eight $\sum_{i=1}^{8} C_i$.

Let

$$\begin{array}{ccc}
Z & \xrightarrow{\epsilon} & \hat{p} \\
\text{Kum}(C^2/\Lambda') & \xrightarrow{\hat{p}} & \text{Kum}(C^2/\Lambda)
\end{array}$$

be the resolution of the map $p$. Then $\hat{p}$ is the double cover of $\text{Kum}(C^2/\Lambda)$ branched along $\sum_{i=1}^{8} C_i$. So $\hat{p} = \hat{p}_X$ and $p = p_X$. $lacksquare$

Note that the composition map $C^2/\Lambda' \xrightarrow{q} C^2/\Lambda \xrightarrow{t} C^2/\Lambda$, where $t$ is any translation by an element in $A_2 - H$, induces the rational double cover

$$\text{Kum}(C^2/\Lambda') \xrightarrow{p'} \text{Kum}(C^2/\Lambda)$$

corresponding to $\sum_{i=1}^{8} C_i'$. Consequently the even eight’s $\sum_{i=1}^{8} C_i$ and $\sum_{i=1}^{8} C_i'$ give rise to the same Kummer surface. We conclude that there exists exactly fifteen Kummer surfaces obtained from a set of sixteen disjoint smooth rational curves on a Kummer surface.
5.3 Geometric Constructions

Let $Y$ be a Jacobian Kummer surface and let $E_0, E_{12}, \cdots, E_{56}$ be sixteen disjoint smooth rational curves on it as in section 2.3.

**Proposition V.6.** Let $Y$ be a general Kummer surface. The fifteen even eight obtained from $E_0, E_{12}, \cdots, E_{56}$ that do not contain $E_0$ are of the form

$$2(L - E_0) - 2(C_{1i} + C_{1j}) - 2E_{ij}, \quad 1 \leq i < j \leq 6.$$ 

The Kummer surface $\text{Kum}(B)$ associated to such an even eight satisfies

$$T_{\text{Kum}(B)} \simeq U(2) \oplus U(2) \oplus (-8).$$

**Proof.** By Theorem II.25, if $i \neq j$ then

$$2(L - E_0) - 2E_{ij} - 2(C_{1i} + C_{1j}) = E_{1i} + \cdots + \hat{E}_{ij} + \cdots E_{i6} + E_{1j} + \cdots + \hat{E}_{ij} + E_{j6}.$$ 

Therefore $2(L - E_0) - 2E_{ij} - 2(C_{1i} + C_{1j})$ is an even eight not containing $E_0$. As there are exactly fifteen choices for $i$ and $j$, we obtain this way all of the possible even eight. Let $\text{Kum}(B)$ be the Kummer surface obtained by taking the double cover branched along such an even eight. By Theorem V.5, the polarization of $B$ is of type $(1, 2)$, i.e. $T_B \simeq U \oplus U \oplus (-4)$ which implies by Proposition II.21 that $T_{\text{Kum}(B)} \simeq U(2) \oplus U(2) \oplus (-8)$. 

**Remark V.7.** Note that the abelian surface $B$ is not principally polarized even though the Kummer surface $Y$ is associated to a principally polarized abelian surface.

The fifteen even eight in Proposition V.6 define a decomposition of the sextic $S$ (see figure 2.2) into a quartic and a conic. For instance, the even eight

$$\Delta_{12} = E_{13} + E_{14} + E_{15} + E_{16} + E_{23} + E_{24} + E_{25} + E_{26}$$

is equal to

$$2(L - E_0) - 2E_{12} - 2(C_0 + C_{12})$$

thus it corresponds to the decomposition of the sextic $S$ into the conic $C = l_1 + l_2$ and the residual quartic $Q = l_3 + l_4 + l_5 + l_6$.

**Theorem V.8.** The rational double cover associated to $\Delta_{12}$, $\text{Kum}(B) \dashrightarrow Y$ decomposes as

$$\xymatrix{ \text{Kum}(B) \ar[r]^\phi \ar[d]_\tau & T \ar[d]^\zeta \\ Y \ar[r]^\phi & \mathbb{P}^2 }$$

where $\phi$ is the canonical resolution of the double cover of $\mathbb{P}^2$ branched along $S$. The maps $\zeta$ and $\phi$ are the canonical resolutions of the double covers branched along $Q$ and $\zeta^*(C)$ respectively.
Proof. Consider the pencil of lines passing through the point \( p_{12} \) in \( \mathbb{P}^2 \). Its preimage in \( Y \) defines an elliptic fibration, given by the divisor class

\[ F = L - E_0 - E_{12}. \]

The divisor

\[ F_1 = E_{15} + E_{16} + 2C_0 + E_{13} + E_{14}, \quad \text{and} \quad F_2 = E_{25} + E_{26} + 2C_{12} + E_{23} + E_{14} \]

define two fibers of type \( I_0^* \). The six rational curves \( E_{45}, E_{46}, E_{35}, E_{36}, E_{34}, E_{56} \) are components of six \( I_2 \) fibers. The even eight \( \Delta_{12} \) satisfies

\[ \Delta_{12} \sim F_1 + F_2 - 2(C_0 + C_{12}). \]

By Theorem V.5, the associated double cover is a Kummer surface, \( \text{Kum}(B) \). Moreover the six \( I_2 \) fibers split under the cover and define twelve \( I_2 \) fibers of the elliptic fibration on \( \text{Kum}(B) \) defined by \( \tau^* F \).

On the other hand, let \( T_0 \to \mathbb{P}^2 \) be the double cover of \( \mathbb{P}^2 \) ramified over the reducible quartic \( Q \), then its canonical resolution induces the diagram

\[
\begin{array}{ccc}
T & \xrightarrow{\zeta} & T_0 \\
\downarrow & & \downarrow \\
\tilde{\mathbb{P}}^2 & \to & \mathbb{P}^2
\end{array}
\]

where \( \tilde{\mathbb{P}}^2 \to \mathbb{P}^2 \) is the blowup of \( \mathbb{P}^2 \) at the six singular points of \( Q \). \( T \) is a non-minimal rational surface containing six disjoint smooth rational curves. Indeed by Hurwitz formula, the canonical divisor of \( T \) is given by

\[ K_T = \zeta^*(K_{\mathbb{P}^2} + \frac{1}{2}(l_3 + l_4 + l_5 + l_6)) = -\zeta^*(H) \]

where \( H \) is a hyperplane section. Thus \( K_T^2 = 2, H^2 = 2 \) and \( P_2(T) = 0 \). Denote by \( \tilde{Q} \) the proper transform of \( Q \) in \( T \). Using the additivity of the topological Euler characteristic and the Noether formula, we have

\[ e(T) = e(T - \tilde{Q}) + e(\tilde{Q}) = 10 \Rightarrow \chi(\mathcal{O}_T) = 1 \Rightarrow q(T) = 0 \]

By Castelnuovo’s Rationality Criterion \( T \) is a rational surface. In fact, it can be shown that \( T \) is a weak del Pezzo surface of degree two, i.e. the blow up of \( \mathbb{P}^2 \) at seven points with nef canonical divisor. Indeed the preimages of the four lines \( l_3, l_4, l_5 \) and \( l_6 \) and the preimages of the three “diagonals” of the complete quadrangle formed by \( l_3, l_4, l_5, l_6 \) may be successively blown down to a projective plane.

Consider the following curves of \( T \)

\[ \zeta^*(C) = \zeta^*(l_1 + l_2) = E_1 + E_2, \]

where \( E_1 \) and \( E_2 \) are smooth elliptic curves.
and
\[ \zeta^*(W) = W_1 + W_2 \] where \( W_1 \) and \( W_2 \) are smooth rational curves with the following intersection properties:
\[ E_i^2 = 2, \quad W_i^2 = 0, \quad E_1 \cdot E_2 = 2, \quad W_1 \cdot W_2 = 4, \quad W_i \cdot E_j = 2 \quad \text{for} \quad i \neq j. \]

(recall that \( W \) is the plane conic tangent to the six lines \( l_1, \ldots, l_6 \)). The linear system \( |E_1| \) defines an elliptic fibration on \( T \) with six singular fibers of type \( I_2 \). Take the double cover branched along the two fibers \( E_1 + E_2 \in 2\text{Pic}(T) \). It induces the canonical resolution commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi} & X_0 \\
\downarrow & & \downarrow \\
\tilde{T} & \xrightarrow{\phi_0} & T
\end{array}
\]

where \( \tilde{T} \to T \) is the blowup of \( T \) at the two singular points of \( E_1 + E_2 \).

Claim: \( X \) is a Kummer surface.

Proof of the Claim: Clearly \( K_X = \varphi^*(\zeta^*(-H) + \frac{1}{2}(E_1 + E_2)) = \mathcal{O}_X \).

1) The pullback by \( \varphi \) of the six exceptional curves on \( T \) define twelve disjoint smooth rational curves on \( X \).

2) The two exceptional curves of \( X \) give two more rational curves disjoint from 1).

3) Let \( \varphi^*(W_1) = W'_1 + W''_1 \) and \( \varphi^*(W_2) = W'_2 + W''_2 \) and let \( \sigma \) be the lift on \( X \) of the covering involution of \( \zeta \), then \( \sigma(W'_1) = W''_2 \) or \( \sigma(W'_1) = W'_2 \). Without loss of generality, we can assume that \( \sigma(W'_1) = W'_2 \) and hence get the following intersection numbers
\[ W''_i^2 = W''_i = -2, \quad W'_i \cdot W''_i = 2 \quad \text{for} \quad i = 1, 2 \]
\[ W'_1 \cdot W'_2 = W''_1 \cdot W''_2 = 4 \quad \text{and} \quad W'_1 \cdot W''_2 = W''_1 \cdot W'_2 = 0. \]

One easily checks that \( W'_1 \) and \( W''_2 \) do not intersect the fourteen curves from 1) and 2).

In particular, the \( K3 \) surface \( X \) contains sixteen disjoint smooth rational curves. Consequently \( X \) is a Kummer surface. Moreover, the surface \( X \) contains an elliptic fibration with twelve \( I_2 \) fibers. It also admits two non symplectic involutions \( \theta \) and \( \sigma \) where \( \theta \) is the covering involution of the map \( \varphi \) and \( \sigma \) is the lift of the covering involution of \( \zeta \) on \( T \) encountered earlier. The composition \( \iota = \varphi \circ \sigma \) defines a symplectic involution on \( X \) whose quotient is a \( K3 \) surface admitting an elliptic fibration with singular fibers identical to the one defined by \( F \) on \( Y \). In fact, we can now recover sixteen disjoint rational curves on the quotient and conclude that it is our original general Kummer surface \( Y \) and that \( X \simeq \text{Kum}(B) \).
CHAPTER VI

Open Problems

This section gives a brief account of the open problems related to this dissertation. Morally, it reflects the wish to unravel the intriguing connections that these families of $K3$ surfaces seem to draw between numerous areas of algebraic geometry.

- The table of section 3.3 lists all the types of $K3$ surfaces covering a fixed Kummer surface, but it does not solve the question of finding how many $K3$ surfaces exist in each type. Although we know that this number is finite, we have not been able to give a clear counting formula. We suspect that the answer to this question amounts to counting non-isomorphic sublattices of a given lattice.

- From a more geometrical aspect, it would be very nice to have a complete geometric description of all the $K3$ surfaces described in the table of the section 3.3. In particular, we would like to identify the $K3$ surface associated to a well known even eight obtained as a combination of nodes and tropes.

- Results in section 4.4 suggests a relation between all the $K3$ surfaces constructed in chapter IV and the Brauer group of the $K3$ surface with Shioda-Inose structure. It would be very illuminating to find the exact nature of this relation and to connect it to the theorem of Hosono, Liang, Oguiso and Yau describing all the Abelian surfaces associated to a fixed Kummer surface ([HLOY03]).

- Using the language of moduli spaces, we wish to study the boundary divisors of the moduli spaces of $K3$ surfaces with symplectic involutions and quotient birational to a Kummer surface and see its connection to the degeneration of the different sextics encountered all along this thesis.
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ABSTRACT

Even Eight on a Kummer Surface

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This thesis addresses the problem of finding all the double covers of a given Kummer surface branched along an even eight. An even eight is defined to be a set of eight smooth disjoint rational curves whose sum determine an even class in the Picard group of the Kummer surface. By a well known result of Nikulin, this problem is equivalent to the classification of all the $K3$ surfaces admitting a symplectic involution such that the corresponding quotient is birational to a given Kummer surface.

This dissertation is divided into two parts. In the first part, we give a complete classification of all such $K3$ surfaces by giving a criterion on their transcendental lattice. We show that up to isomorphism those $K3$ surfaces form a finite set. When the Kummer surface is general, we give a list of all the possible transcendental lattices of the corresponding $K3$ surfaces.

The second part deals with the geometrical interpretation of some of the $K3$ surfaces obtained in the list above. We give new double plane models of the general Kummer surface and show that those $K3$ surfaces correspond to the decomposition of the branch locus of these new models. This correspondence allows us to give a full geometrical description of the Picard group of the $K3$ surfaces. Finally, we show that there exist in our list pairs of $K3$ surfaces where one is a Jacobian elliptic fibration and the other is a torsor over this Jacobian. This draws a connection between the already known classification of abelian surfaces arising as the double cover of a Kummer surface and our classification.