Quantum minimal surfaces and Kähler noncommutative geometry

(based on 1903.10792 with J. Arnlind, J. Hoppe, and on 2003.03171 with G. Bhattacharya)
Action functional (A. Schild, 1976): for a map \( \varphi : (\Sigma^2, \omega) \rightarrow (X^n, g) \) define

\[
S(\varphi) := \int_{\Sigma^2} |\wedge^2 T_\varphi(\omega^{-1})|^2 \cdot \omega
\]

If we replace exponent \(|?|^2\) by \(|?|\) \(\sim\) Goto-Nambu action = the usual area with respect to the induced metric (does not depend on \(\omega\)).

Critical points (solutions of the Euler-Lagrange equation):

\[
\begin{cases}
\text{non-degenerate:} & \varphi(\Sigma^2) \text{ is a minimal surface, } \omega^{\times const} \text{ volume(induced metric)}, \quad S(\varphi) > 0 \\
\text{degenerate:} & \dim \varphi(\Sigma^2) \leq 1 \text{ arbitrary map with 0- or 1-dimensional image}, \quad S(\varphi) = 0
\end{cases}
\]

Example: \((X^n, g)\) is the usual Euclidean space \(\mathbb{R}^n\). Map \(\varphi = \vec{x} = (x_1, \ldots, x_n), \quad x_i \in C^\infty(\Sigma^2)\).

\[
S(\vec{x}) = \int_{\Sigma^2} \sum_{i<j} \{x_i, x_j\}^2
\]

\[
\frac{\delta S}{\delta \varphi} = 0 \iff \forall i \quad \sum_j \{x_j, \{x_j, x_i\}\} = 0
\]
Schild action

\[ S(\varphi) = \int_{\Sigma^2} |\wedge^2 T_{\varphi}(\omega^{-1})|^2 \cdot \omega \]

is homogeneous in \( \omega \), of homogeneity degree \(-1\). Define the **normalized** Schild functional as

\[ S_{\text{norm}}(\varphi) := \left( \int_{\Sigma^2} \omega \right) \cdot \left( \int_{\Sigma^2} |\wedge^2 T_{\varphi}(\omega^{-1})|^2 \cdot \omega \right) \]

It is invariant under rescaling \( \omega \mapsto \lambda \omega, \; \lambda \in \mathbb{R}_{>0} \). Critical values of \( S_{\text{norm}} \) are

\[ \{0\} \cup \{A^2 \mid A = \text{area of a non-trivial minimal surface}\} \]

Our next goal: define a **quantum analog** of the normalized Schild action...
Quantization

\( \hbar \to 0 \)

\((M, \omega) : \text{compact symplectic} \quad \leadsto \quad \mathcal{H} = \mathcal{H}_\hbar : \text{finite-dimensional Hilbert space/} \mathbb{C} \)

\(C^\infty\)-manifold of dimension \(2d\)

\( f \in C^\infty_\mathbb{R}(M), \; f = \bar{f} \quad \leadsto \quad \text{self-adjoint operator} \; \hat{f} = \hat{f}^\dagger \in \text{End}(\mathcal{H}) \)

\( g \in C^\infty_\mathbb{C}(X), \; |u| = 1 \quad \leadsto \quad \text{unitary operator} \; \hat{u} \in \text{Aut}(\mathcal{H}) \)

\( \hat{u}_1 \cdot \hat{u}_2 = \hat{u}_1 \hat{u}_2 \cdot \exp \left( \frac{i\hbar}{2} \frac{\{u_1,u_2\}}{u_1 u_2} + O(\hbar^2) \right) \)

\( \frac{i\hbar}{2} \{ \log u_1, \log u_2 \} \)

if \( d = 1, M = \Sigma^2 \)

\( \int_{\Sigma^2} f \cdot \omega = \int_M f \cdot \frac{\omega^d}{d!} \quad \leadsto \quad (2\pi\hbar)^d \text{Tr}_\mathcal{H} \hat{f} + O(\hbar) \)

\( \{ \cdot, \cdot \} \quad \leadsto \quad \frac{1}{i\hbar} [\cdot, \cdot] + O(\hbar) \)
Example: \( d = 1, \quad M = \mathbb{R}/2\pi \mathbb{Z} \times \mathbb{R}/2\pi \mathbb{Z}, \omega = \frac{1}{2\pi} d\theta_1 \wedge d\theta_2, \quad \theta_1, \theta_2 \in \mathbb{R}/2\pi \mathbb{Z}. \)

\( \hbar = \frac{1}{N}, \quad N = 1, 2, \ldots, \) Hilbert space \( \mathcal{H} := \mathbb{C}^N. \) Operators corresponding to \( u_1 = e^{i\theta_1}, u_2 = e^{i\theta_2} \) are

\[
\hat{u}_1 = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}, \quad \hat{u}_2 = \begin{pmatrix}
1 \\
e^{2\pi i/N} \\
\vdots \\
e^{2\pi i(N-1)/N}
\end{pmatrix}
\]

**General construction** (Berezin-Toeplitz quantization): assume \( \frac{1}{2\pi} [\omega] = c_1(\mathcal{L}) \in \text{image} \left( H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R}) \right) \), where \( \mathcal{L} \) is a complex line bundle with unitary connection, curvature form \( = i\omega. \) Choose an almost-complex structure on \( M \) compatible with \( \omega. \) Then for \( \hbar = \frac{1}{N} \) where \( N \gg 1 \) is an integer, define \( \mathcal{H}_\hbar \) as the kernel (the cokernel=0 for large \( N \))

\[
\mathcal{H}_\hbar := \text{Ker} \left( \bar{\partial} + \partial^\dagger : \Gamma \left( M, \mathcal{L}^\otimes N \otimes \Omega^{0, \text{even}} \right) \to \Gamma \left( M, \mathcal{L}^\otimes N \otimes \Omega^{0, \text{odd}} \right) \right)
\]

Operator \( \hat{f} = \hat{f}_\hbar \) for \( f \in C_\infty^R(M) \) is defined as the composition

\[
\mathcal{H}_\hbar \hookrightarrow \Gamma \left( M, \mathcal{L}^\otimes N \otimes \Omega^{0, \text{even}} \right) \xrightarrow{f \times \cdot} \Gamma \left( M, \mathcal{L}^\otimes N \otimes \Omega^{0, \text{even}} \right) \xrightarrow{\text{orthogonal projection}} \mathcal{H}_\hbar
\]
Normalized Schild functional, case \( \dim M = 2d = 2 \), \( N = \dim \mathcal{H}_\hbar \), \( X_i = X_i^\dagger = \hat{x}_i \), \( 1 \leq i \leq n \):

\[
\int_M \omega \cdot \int_M \sum_{i < j} \{x_i, x_j\}^2 \omega \leadsto 2\pi \hbar \text{Tr} \mathbf{1}_{N \times N} \cdot 2\pi \hbar \text{Tr} \sum_{i < j} \left( \frac{1}{i\hbar} [X_i, X_j] \right)^2 = -(2\pi)^2 N \cdot \text{Tr} \sum_{i < j} [X_i, X_j]^2
\]

Euler-Lagrange equation
\[
\sum_{j} [X_j, [X_j, X_i]] = 0
\]

Problem: \( \exists \) \textit{compact} minimal surfaces in \( \mathbb{R}^n \). Various ways to deal with it:

1. Impose constraints, e.g. to lie on a sphere: \( \sum_i x_i^2 = 1 \leadsto \sum_i X_i^2 = \mathbf{1}_{N \times N} \), etc.
2. Map to \( U(1)^n \) with constant flat metric \( g = g^t > 0 \leadsto \) collection of \( n \) unitary operators \( (\mathcal{U}_i)_{1 \leq i \leq n}, \mathcal{U}_i \in U(N) \forall i \)

\[
S_{\text{quantum}} = N \cdot \sum_{i < j} \det_{2 \times 2} \begin{pmatrix} g_{ii} & g_{ij} \\ g_{ji} & g_{jj} \end{pmatrix} \text{Tr} \left( 2 \cdot \mathbf{1}_{N \times N} - \mathcal{U}_i \mathcal{U}_j \mathcal{U}_i^{-1} \mathcal{U}_j^{-1} - \mathcal{U}_j \mathcal{U}_i \mathcal{U}_j^{-1} \mathcal{U}_i^{-1} \right)
\]

\textbf{Conjecture:} critical values of quantum Shild action \( \text{Tr} \cdot \text{Tr} \), on \textit{indecomposable} solutions of Euler-Lagrange equation \( \longrightarrow \) \textit{nondegenerate} critical values of the normalized Schild action \( \int \cdot \int \)

(= squares of areas of minimal surfaces).
Side remark: for a map $\varphi : \Sigma^2 \to X^d$ one can fix the homology class $\varphi_*[\Sigma^2] \in H_2(X, \mathbb{Z})$. E.g., when $X$ is Kähler, then one has a lower bound: $\text{Area}(\varphi(\Sigma^2)) \geq |\langle \varphi_*[\Sigma^2], [\omega_{X}^{1,1}] \rangle|$ which is saturated for holomorphic curves (basic example of calibrated geometry).

Analogs of degree for “quantum maps”:

1. $X = U(1) \times \cdots \times U(1)$, a “quantum map” is $(\mathcal{U}_1 = \hat{\mu}_1, \ldots, \mathcal{U}_n = \hat{\mu}_n), \mathcal{U}_i \in U(N) \ \forall i$ $n$ times

What is “homology” class in $H_2(X, \mathbb{Z}) = \mathbb{Z}^{n(n-1)/2}$? Answer: component $(i, j)$ for $i < j$ is

$$\deg_{(i,j)} \varphi = \frac{1}{(2\pi i)^2} \int \left\{ \log u_i, \log u_j \right\} \omega \approx \frac{1}{2\pi i} \text{Tr} \left( \mathcal{U}_i \mathcal{U}_j \mathcal{U}_i^{-1} \mathcal{U}_j^{-1} - \mathbf{1}_{N \times N} \right)$$

**Lemma**: close to an integer if $|\mathcal{U}_i \mathcal{U}_j - \mathcal{U}_j \mathcal{U}_i| < \frac{c}{N}$

2. $X = S^2 : \ x_1^2 + x_2^2 + x_3^2 = 1 \subset \mathbb{R}^3$, $\deg \varphi = \frac{3}{4\pi} \int x_1 \{x_2, x_3\} \omega$

**Lemma**: $\forall X_1, X_2, X_3 : X_1^2 + X_2 + X_3^2 = \mathbf{1}_{N \times N}$ $|X_i X_j - X_j X_i| < \frac{c}{N}, \quad X_i^\dagger = X_i, \quad (N \to +\infty)$

$$\implies \frac{3i}{2} \text{Tr} X_1 [X_2, X_3] \text{ is close to an integer}$$
Non-compact case

Minimal surfaces in $\mathbb{R}^n$ of “finite type” (up to translation)

\[ \uparrow \text{ Weierstrass parametrization: } dx_i = \text{Re } \alpha_i \]

Punctured complex algebraic curve $C = \overline{C} - \{\text{punctures}\}$ and a collection $\alpha_1, \ldots, \alpha_n \in \Gamma(C, K_C)$ of holomorphic 1-forms, satisfying

1. $\forall i$ all periods of $\alpha_i$ are purely imaginary,

2. $\sum_i \alpha_i \otimes^2 = 0 \in \Gamma(C, K_C^{\otimes^2})$.

This is (almost) a question of algebraic geometry.

Back to the compact case: minimal surfaces in flat tori, - imaginary parts of periods are specified.

Quantum analog: $\infty$-dimensional Hilbert space $\mathcal{H}$, a completion of $\Gamma(C, K_C^{\otimes^{1/2}})$, 

Euler-Lagrange equations:

\[ \forall i \sum_j [X_j, [X_j, X_i]] = 0, \quad X_i = X_i^\dagger \text{ are unbounded operators in } \mathcal{H} \]

Problem: precise formulation of “boundary conditions at infinity”.

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Calibrated geometry *(supersymmetry)*

Equations of 2-nd order $\iff$ Equations of 1-st order “harmonicity” *much simpler*

Explanation: BPS inequality

Mass $\geq |\text{Topological charge}|$

Examples:

1. In Kähler manifold $X$, all complex submanifolds are minimal
   
   1-st order equations
   (Cauchy–Riemann)

2. On Kähler manifold $X$, all holomorphic vector bundles, endowed with Hermitian norm satisfying

   Hermitian Yang-Mills equations give solutions of the usual Yang-Mills equations

   also 1-st order equations on connection

   2-nd order equations on connection
**Yang-Mills algebra** (A.Connes, M.Dubois-Violette)

\[ YM_n := \mathbb{C}\langle X_1, \ldots, X_n \rangle / \forall i \sum_j [X_j, [X_j, X_i]] = 0 \]

with \(*\)-structure: \( X_i^* = X_i \) \( \forall i \).

**Explanation of the name**: “constant” (i.e. translationally-invariant) connection on the trivial bundle on \( \mathbb{R}^n \)

\[ \nabla = d + \sqrt{-1} \sum_i X_i dx_i \]  
(here \( x_i \) are coordinates)

satisfies YM equations \( \Leftrightarrow \) a finite-dimensional \(*\)-representation of \( YM_n \).

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**Hermitian Yang-Mills algebra** \( (n = 2m) \)

\[ HYM_m := \mathbb{C}\langle Z_1, \ldots, Z_m, Z_1^*, \ldots, Z_m^* \rangle / \text{relations} \]

\[ \forall i, j, [Z_i, Z_j] = 0 = [Z_i^*, Z_j^*], \sum_{k=1}^m [Z_k^*, Z_k] = 0 \]

Generalization: \([Z_i, Z_j] = c_{ij} = -c_{ji} \in \mathbb{C}, \ [Z_i^*, Z_j^*] = -\bar{c}_{ij}, \sum_{k=1}^m [Z_k^*, Z_k] = c \in \mathbb{R}\)

translational-invariant solutions of HYM in \( \mathbb{C}^m \) \( \Leftrightarrow \) finite-dimensional \(*\)-representations of \( HYM_m \).

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Correspondence: \[
\begin{align*}
Z_k &= X_k + \sqrt{-1}X_{m+k} \\
Z_k^* &= X_k - \sqrt{-1}X_{m+k}
\end{align*}
\]

classical limit: \[
\begin{align*}
\{z_i, z_j\} &= 0 = \{\bar{z}_i, \bar{z}_j\}, \quad \sum_k \{\bar{z}_k, z_k\} = 0
\end{align*}
\]
Free Hermitian Yang-Mills algebra = King’s algebra

\[ K_m := \mathbb{C}\langle T_1, \ldots, T_m, T_1^*, \ldots, T_m^* \rangle / \sum_{i=1}^{m} [T_i^*, T_i] = 0 \]

only one relation!

There are two morphisms of $\ast$-algebras:

\[ K_m \rightarrow HYM_m \rightarrow YM_{2m} \]

**Theorem** (A.King, particular case):

finite-dimensional $\ast$-representations of $K_m$, up to a $\ast$-isomorphism

\[ 1:1 \leftrightarrow \]

finite-dimensional **semi-simple** representations of the free algebra $\mathbb{C}\langle T_1, \ldots, T_m \rangle$, up to an isomorphism.
Explanation: representations of $\mathbb{C}\langle T_1, \ldots, T_m \rangle \rightarrow \text{End}(\mathbb{C}^N)$/ iso
$\iff$ a collection of $N \times N$-matrices $(M_1, \ldots, M_m) \in \mathbb{C}^{mN^2}$ modulo action of $GL_N(\mathbb{C})$

Fact: a representation is semisimple $\iff$ the corresponding $GL_N(\mathbb{C})$-orbit is closed.

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**Geometric Invariant Theory (Mumford, Kempf-Ness)**

For $\forall$ reductive group $G/\mathbb{C}$ and a finite-dimensional representation $V$: choose a maximal compact subgroup $K \subset G(\mathbb{C})$ and $K$-invariant Hermitian norm on $V$, hence $\rho(K) \subset U(V)$.

For $v \in V$, an orbit $G \cdot v$ is closed $\iff$ $\mathbb{R}_{\geq 0}$-valued function $g \cdot v \mapsto |g \cdot v|^2$ on the orbit achieves the minimum. Moreover, in this case the set of minima is one $K$-orbit.

The action of $K$ on $V$ considered as a real symplectic manifold, is Hamiltonian, hence gives a moment map $\mu : V \rightarrow \text{Lie}(K)^*$. The closed subset $\mu^{-1}(0) \subset V$ is the set of closests to 0 points of orbits.

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In the special case $G = GL_\mathbb{C}(N)$ and $V = \mathbb{C}^{mN^2}$ the moment map is given by

$$(M_1, \ldots M_m) \mapsto \sqrt{-1} \sum_{i=1}^{m} [M_i^\dagger, M_i]$$
Corollary (of King’s theorem):

for any finitely-generated associative algebra $A/C$ and any finite collection of generators $(a_i)_{i \in I}$:

\[
\{ \text{Finite-dimensional semisimple representations of } A \} / \text{isomorphisms} \\
\overset{1:1}{\iff} \\
\{ \text{Finite-dimensional } \ast\text{-representations of } B = B_{(a_i)_{i \in I}} \} / \ast\text{-isomorphisms}, \text{ where } \\
B_{(a_i)_{i \in I}} := A \ast A^\ast \text{ / relation } \sum_i [a_i^\ast, a_i] = 0
\]

Corollary of corollary: Different choices of finite-collections of generators give different $\ast$-algebras $B = B_{(a_i)_{i \in I}}, B' = B_{(a'_j)_{j \in J}}$ with the same theory of finite-dimensional $\ast$-representations.

Conjecture: an appropriate $\ast$-completion does not depend on the choice of generators.

E.g., I expect that the convex body

\[
\{ \text{Normalized states on } B \} := \{ \tau : B \to \mathbb{C} \mid \tau(1_B) = 1, \quad \tau(b^\ast) = \overline{\tau(b)}, \quad \tau(b^\ast b) \geq 0 \}
\]

does not depend on the choice of $(a_i)_{i \in I}$. It could be non-empty even if $A$ has no non-trivial finite-dimensional representations.
Let us return to finite-dimensional representations.

Any choice of generators \((a_i)_{i \in I}\) gives a function \(H_N = \sum_i \text{Tr} \phi(a_i) \phi(a_i) \geq 0\) on the space

\[ R_N := \{ N \text{-dimensional semisimple representations } \phi : A \to \text{End}(\mathbb{C}^N) \} / \text{iso} \]

\[ \leftarrow \text{singular reduced affine scheme of finite type} \]

Function \(H_N\) is a strictly plurisubharmonic function on \(R_N\), potential of a singular Kähler metric on \(R_N\).

We can think of the collection of metrics \(\sqrt{-1} \partial \bar{\partial} H_N\) on \(R_N\) for all \(N \geq 1\) as a shadow of a "noncommutative Kähler metric on Spec\(A\)" induced from the “flat metric” on the “spectrum” of free algebra via the embedding

\[ \text{“Spec}\ A \hookrightarrow \text{Spec} \, \mathbb{C}\langle a_1, \ldots, a_m \rangle \]

\[ A \hookleftarrow \mathbb{C}\langle a_1, \ldots, a_m \rangle \]

Conjecture about independence of \(*\)-completion \(B_{\text{univ}} \supset B_{(a_i)_{i \in I}}\) on the choice of generators: we have an embedding \(A \hookrightarrow B_{\text{univ}}\) defined up to conjugation by an invertible element \(b \in B_{\text{univ}}\).

It looks as choices of an embedding from this conjugacy class

\[ \iff \]

“noncommutative Kähler metrics” on \(A\) (could serve as a definition?).
Generalization: finite quiver $Q = (Q_0, Q_1, \text{a map } Q_1 \to Q_0 \times Q_0)$, e.g. $\bullet \xrightarrow{\alpha} \bullet \to \bullet$.

A representation of $Q$: vector spaces $E_v$ for any $v \in Q_0$, morphisms $E_{v_1} \to E_{v_2}$ for any arrow $v_1 \xrightarrow{\alpha} v_2$.

Definition: for a given collection of weights $\mu = (\mu_v \in \mathbb{R})_{v \in Q_0}$, a finite-dimensional representation of $Q$ is called $\mu$-stable iff $\sum_v \mu_v \dim E_v = 0$ and for any subrepresentation $0 \subsetneq E' \subsetneq E$ one has $\sum_v \mu_v \dim E'_v < 0$. A representation is called $\mu$-polystable iff its is a sum of $\mu$-stable ones.

Theorem (A.King)

Isomorphism classes of $\mu$-polystable representations

$\iff$

isomorphism classes of $*$-representations of $*$-algebra generated by $\Pi_v = \Pi_v^*$ for $v \in Q_0$ (orthogonal projectors to the summands in $\bigoplus_v E_v$), $T_\alpha, T_\alpha^*$ for $\alpha \in Q_1$ with relations $\Pi_{v_1} \Pi_{v_2} = \delta_{v_1,v_2} \Pi_{v_1}$, $\sum_v \Pi_v = 1$, $T_\alpha = \Pi_{v_2} T_\alpha \Pi_{v_1}$, $T_\alpha^* = \Pi_{v_1} T_\alpha^* \Pi_{v_2}$ for $v_1 \xrightarrow{\alpha} v_2$.

And $\sum_\alpha [T_\alpha^*, T_\alpha] = \sum_v \mu_v \Pi_v$.

Generalization of the conjecture on the uniqueness of $*$-completion.
Finitely-generated associative algebra = finitely-generated free algebra/arbitrary relations

\[ \text{Finitely-generated algebra } \supseteq \mathbb{C}^{Q_0} \text{ } = \text{ path algebra of finite quiver } \mathbb{C}\langle Q\rangle/ \text{ arbitrary relations} \]

Further generalization:

\[ A = D/ \text{ any two-sided ideal, } \quad \text{where} \]

\[ D = D_{\leq 0} + D_{\leq 1} + D_{\leq 1} \otimes_{D_{\leq 0}} D_{\leq 1} + \ldots / \text{ relations } i(a) - a = 0 \ \forall a \in D_{\leq 0} \]

where \( D_{\leq 0} \) is an associative algebra, and \( D_{\leq 1} \) is a bimodule over \( D_{\leq 0} \) endowed with a morphism \( i : D_{\leq 0}^{\text{diag}} \rightarrow D_{\leq 1} \) of bimodules (and then \( D = (D_{\leq 0} \subset D_{\leq 1} \subset \ldots) \) is a filtered algebra).

- **Example 1** (quiver): \( D_{\leq 0} = \mathbb{C}^{Q_0}, D_{\leq 1} = \mathbb{C}^{Q_0} + \mathbb{C}^{Q_1} \), then \( D \) is the path algebra \( \mathbb{C}\langle Q\rangle \).
- **Example 2** ("free differential operators"): \( D_{\leq 0} = C^\infty_\mathbb{C}(X) = \{ \text{differential operators of order 0} \}, D_{\leq 1} = C^\infty_\mathbb{C}(X) + \Gamma(X, T_\mathbb{C}X) = \{ \text{differential operators of order } \leq 1 \} \).

Then a \( D^{\text{free}} \)-module which is finitely-generated as \( C^\infty_\mathbb{C}(X) \)-module, is the same as a vector bundle on \( X \) with a **not neccesarily flat** connection.
Data giving moment map and Kähler metric on the space of isomorphism classes of “polystable” representations

1. $D_{\leq 0}$: a $*$-algebra.

2. $\Omega^1 \in D_{\leq 0} - \text{mod} - D_{\leq 0}$, a bimodule which is finitely-generated projective as a right module, endowed with a derivation $d : D_{\leq 0} \to \Omega^1$, $d(fg) = fg + df \cdot g$.

$\leadsto$ extension of bimodules $0 \to \Omega^1 \to B \to D^{\text{diag}}_{\leq 0} \to 0$

$\leadsto$ define $D_{\leq 1} := \text{Hom}_{\text{mod-}D_{\leq 0}}(B, D_{\leq 0})$, automatically have a morphism $i : D^\text{diag}_{\leq 0} \to D_{\leq 1}$.

3. “(Kähler form)$^{-1}$”: a non-negative pre-Hilbert form $\langle \cdot , \cdot \rangle : \Omega^1 \otimes \bar{\Omega}^1 \to \mathbb{C}$ satisfying $\langle f \cdot \alpha \cdot g , \beta \rangle = \langle \alpha , f^* \cdot \beta \cdot g^* \rangle$.

4. Analog of weights $(\mu_v)_{v \in Q_0}$: a linear map $\mu : D_{\leq 0} \to \mathbb{C}$

satisfying $\mu(f^*) = \mu(f)$, $\mu([f, g]) = \langle df, d^*g \rangle - \langle dg, df^* \rangle$.

Comment: the r.h.s. in $\square$ is a a cyclic 2-cochain, the equation says that it is a coboundary.
in this way one obtains:

- King’s equations
- Hermitian Yang-Mills equations
- Nahm equations
- Bogomolny (a.k.a. monopole) equations
- Hitchin equations
- Vafa-Witten equations
- vortex equations (Álvarez-Cónsul, García-Prada)

(all the above examples: only for the gauge groups $U(N)$)

- noncommutative instantons (Connes-Douglas-Nekrasov-Schwarz), where

\[ D_{\leq 0} = C^\infty(\mathbb{R}^{2m}_{\text{quantum}} = \mathbb{C}^m_{\text{quantum}}) \text{ or } C^\infty(\text{quantum Kahler torus } \mathbb{C}^m_{\text{quantum}}/\mathbb{Z}^{2m}) \]
Quantum $\mathbb{C}^m$

$D_{\leq 0}$ is topologically (or $C^\infty$) generated by variables $z_1, \ldots, z_m, z_1^*, \ldots, z_m^*$ satisfying relations

$[z_i, z_j] = 0 = [z_i^*, z_j^*]$
$[z_i^*, z_j] = \hbar \delta_{ij}$

and has commuting derivations

$\partial_1, \ldots, \partial_m, \bar{\partial}_1, \ldots, \bar{\partial}_m$

$satisfying relations$

$\partial_i (z_j) = \delta_{ij}, \quad \partial_i z_j^* = 0$
$\bar{\partial}_i (z_j) = 0, \quad \bar{\partial}_i z_j^* = \delta_{ij}$

Noncommutative HYM instanton $:=$ finitely-generated projective $D_{\leq 0}$-module $E$ endowed with a $D_{\leq 0}$-valued hermitean form $H : E \otimes_{\mathbb{C}} \overline{E} \to D_{\leq 0}, \quad H(f \cdot \phi_1, g \cdot \phi_2) = f \cdot H(\phi_1, \phi_2) \cdot g^\dagger$

and connection given by covariant derivatives $\nabla_i, \bar{\nabla}_i : E \to E$ satisfying

$[\nabla_i, \nabla_j] = [\bar{\nabla}_i, \bar{\nabla}_j] = 0$
$[\nabla_i, z_j] = [\bar{\nabla}_i, z_j^*] = \delta_{ij}, \quad [\nabla_i, z_j^*] = [\bar{\nabla}_i, z_j] = 0, \quad \sum_i [\bar{\nabla}_i, \nabla_i] = 0$

$H(\bar{\nabla}_i \phi_1, \phi_2) + H(\phi_1, \nabla_i \phi_2) = \bar{\partial}_i H(\phi_1, \phi_2)$ \quad HYM equation
**Construction** (Fuurushi, Nekrasov, ~2000)

**Noncommutative HYM instanton** $\rightsquigarrow$ infinite-dimensional solution of King's equation

Define a state $\tau_\rho : D_{\geq 0} \to \mathbb{C}$ (depends on $\rho > 0$): $\tau_\rho(\prod_i z_i^{k_i} \cdot \prod_i z_i^{l_i}) := \prod_i \delta_{k_i,l_i} \rho^{k_i} k_i!$

$\rightsquigarrow$ pre-Hilbert scalar product on $E$: $\langle \phi_1, \phi_2 \rangle := \tau_\rho H(\phi_1, \phi_2)$

$\rightsquigarrow$ Hilbert space completion $\mathcal{H}_{big}$ (morally $L_2$-sections of $E$)

Define $\forall i \ Z_i := z_i - \rho \overline{\nabla}_i \quad \text{calculation} \quad Z_i^\dagger = \rho \nabla_i \quad \Rightarrow \quad \sum_i [Z_i^\dagger, Z_i] = \rho \cdot m \cdot \text{id}_{\mathcal{H}_{big}}$ (King's equation!)

**Non-trivial fact:** operators $Z_i, Z_i^\dagger$ preserve subspace $\mathcal{H} := \cap_i \text{Ker} \overline{\nabla}_i \subset \mathcal{H}_{big}$, hence operators obtained by the restriction $(Z_i|_\mathcal{H})_{1 \leq i \leq m}$ also obey King's equation. This is the desired solution.

Subspace $\mathcal{H}$ is analogous to the space of holomorphic sections, and is a Hilbert completion of a finitely-generated module $V$ over $\mathbb{C}[Z_1, \ldots, Z_m] \simeq \mathbb{C}[z_1, \ldots, z_m]$.

In the original examples in noncommutative HYM instantons, module $V$ could corresponds to a usual stable vector bundle on $\mathbb{C}P^2$ trivialized at infinity $\mathbb{C}P^1_\infty = \mathbb{C}P^2 - \mathbb{C}^2$ (as in ADHM theory), and also to a torsion free sheaf (which is not a reflexive sheaf) like, e.g., the ideal of a finite subscheme in $\mathbb{C}^2$. 
Relation with quantum minimal surfaces

Example: $m = 2$, $V = \mathbb{C}[z_1, z_2]/f \cdot \mathbb{C}[z_1, z_2]$ where $f = f(z_1, z_2) \in \mathbb{C}[z_1, z_2]$ is the polynomial defining an affine curve $\Sigma$ by equation $f(z_1, z_2) = 0$. This is not a noncommutative HYM instanton, as $V$ is a torsion module, (the property which is just opposite to be torsion-free).

Still, the infinite-dimensional King's equation for a pre-Hilbert norm on $V$ makes sense, and it was considered long time ago (~1998) by L.Cornalba and W.Taylor IV. They conjectured that the solution exists, and studied few examples. By our previous considerations, such solutions are noncompact quantum minimal surfaces in $\mathbb{R}_{\text{classical}}^{2m}$ which happen to be “supersymmetric”, i.e. complex curves.

Example: complex hyperbola $\Sigma = \mathbb{C}^\times \hookrightarrow \mathbb{C}^2$, $z \mapsto (z, z^{-1})$, i.e. $f = z_1 z_2 - 1$.

Scalar product on $V = \mathbb{C}[z, z^{-1}]$ is $\langle z^n, z^m \rangle = \delta_{nm} c_n$ where $\frac{c_{n+1}}{c_n} = \frac{n\rho + \sqrt{n^2 \rho^2 + 4}}{2} \quad \forall n \in \mathbb{Z}$.

Conjecture (need to be formulated precisely):

for any “stable” (?) finitely-generated module $V$ over $\mathbb{C}[Z_1, \ldots, Z_m]$ with support of pure dimension $0 \leq d \leq m$, and an appropriate condition at infinity (?)*, there exists a unique (up to scalar) pre-Hilbert norm on $V$ such that

$$\sum_i [Z_i^\dagger, Z_i] = \rho \cdot d \cdot \text{id}_V$$

A guess for (?)*: a solution of the classical HYM equations at $\overline{\text{Supp}V} \cap \mathbb{C}P_{\infty}^{m-1}$, Fubini-Study metric.
Thank you!