An index theorem on the tempered dual of a real reductive Lie group

Xiang Tang

Washington University in St. Louis

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Outline

In this talk, we will report our study on the geometry of the tempered dual of a (real reductive) Lie group $G$. As an application, we will present an index theorem for proper cocompact $G$ actions.
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1. The example of $SL(2, \mathbb{R})$
2. Geometry of the tempered dual
3. An index theorem
This talk is based on several joint works with Peter Hochs, Markus Pflaum, Hessel Posthuma, and Yanli Song.
Let $SL(2, \mathbb{R})$ be the Lie group of $2 \times 2$ real matrices with determinant being 1, e.g.

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Let $X$ be the quotient $SL(2, \mathbb{R})/SO(2)$, which can be identified with the Poincaré disk.

The left action of $SL(2, \mathbb{R})$ on $X$ is proper, and $X$ is equipped with an $SL(2, \mathbb{R})$-invariant Kähler structure.
We look at the Dolbeault complex on $X$, e.g.

$$\overline{\partial} : \Omega^{0,0}(X) \rightarrow \Omega^{0,1}(X).$$
Euler characteristic

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The differential operator $\bar{\partial}$ is invariant with respect to the $SL(2, \mathbb{R})$ action. The (co)kernel of $\bar{\partial}$ is naturally equipped with an $SL(2, \mathbb{R})$-representation. Therefore, $H^{0,0}(X)$ and $H^{0,1}(X)$ are also equipped with $SL(2, \mathbb{R})$-representations.
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**Question**

Understand the Euler characteristic, $[H^{0,0}(X)] - [H^{0,1}(X)]$?
Line bundles on $X$

Let $\mathbb{C}_n$ be the 1-dimensional complex vector space equipped with an $SO(2)$ representation of weight $n$, e.g.

$$\rho_n : \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \mapsto T^n_\theta,$$

$$T^n_\theta : \mathbb{C}_n \to \mathbb{C}_n, \quad T^n_\theta(z) := \exp(2\pi \sqrt{-1} n \theta) z.$$
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Define $\tilde{V}_n := SL(2, \mathbb{R}) \times SO(2) \mathbb{C}_n$, an $SL(2, \mathbb{R})$-equivariant line bundle over $X$. 

**Question**

Understand the index of $\partial_n$, $[\ker(\partial_n)] - [\text{coker}(\partial_n)]$. 

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Index theory on the tempered dual of a real reductive Lie group
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With an appropriate connection on $\tilde{V}_n$, we obtain the following generalization of the Dolbeault operator

$$\overline{\partial}_n : \Omega^{0,0}(X, \tilde{V}_n) \to \Omega^{0,1}(X, \tilde{V}_n).$$
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**Question**

Understand the index of $\overline{\partial}_n$, $[\ker(\overline{\partial}_n)] - [\coker(\overline{\partial}_n)]$. 
The example of $SL(2, \mathbb{R})$

Geometry of the tempered dual

Index theory

Unitary representation

- We can choose $SL(2, \mathbb{R})$-invariant Hermitian metrics on $T^*X$ and $\tilde{V}_n$ so that

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  \bar{\partial}_n : \mathcal{L}^2\left(\Omega^{0,0}(X, \tilde{V}_n)\right) \rightarrow \mathcal{L}^2\left(\Omega^{0,1}(X, \tilde{V}_n)\right)
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  is an unbounded closed operator.
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- The kernel $\ker(\bar{\partial}_n)$ is a closed $SL(2, \mathbb{R})$-invariant subspace, and therefore is a unitary $SL(2, \mathbb{R})$ representation.
- Define $\text{ind}(\bar{\partial}_n) := [\ker(\bar{\partial}_n)] - [\text{coker}(\bar{\partial}_n)] \in \text{Rep}(SL(2, \mathbb{R}))$. 
Regular representations

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$$L_f(\xi)(g) := \int_G f(h)\xi(h^{-1}g)d\mu(h).$$
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Observation : $L_f$ is a bounded linear operator on $L^2(G)$.

Definition

- The (reduced) $C^*$-algebra $C^*_r(G)$ is the norm closed $*$-subalgebra of $B(L^2(G))$ generated by $L_f$ for all $f \in C_c(G)$.
- The Harish-Chandra Schwartz algebra $\mathcal{C}(G)$ is a subalgebra of $C^*_r(G)$ consisting of functions on $G$ with rapid decay derivatives.
Let $K$ be a maximal compact subgroup of $G$, and $X = G/K$. 
Index

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$$D_\mu : \Gamma(X, S^+ \otimes \tilde{V}_\mu) \to \Gamma(X, S^- \otimes \tilde{V}_\mu).$$

**Definition**

The index of the operator $D_\mu$ is the element

$$\text{Ind}(D_\mu) := [\ker(D_\mu)] - [\text{coker}(D_\mu)] \in K_0(C^*_r(G)).$$
Definition

The tempered dual \( \hat{G}_\lambda \) of \( G \) is the space of isomorphism classes of irreducible unitary representations of \( C^*_r(G) \) equipped with the Fell topology (hull-kernel topology).
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Example

When $G = \mathbb{R}$, $\hat{\mathbb{R}}_\lambda = \mathbb{R}$, and $C^*_r(\mathbb{R}) = C_0(\mathbb{R})$. 
The example of $SL(2, \mathbb{R})$

**Geometry of the tempered dual**

**Index theory**

### Tempered dual

**Definition**

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**Example**

When $G$ is compact, $C^*_r(G) = \bigoplus_\mu \text{End}(V_\mu)$, where $V_\mu$ runs through isomorphism classes of finite dimensional irreducible representations of $G$, and $\hat{G}_\lambda$ is the disjoint union of isomorphism classes of irreducible representations.
The example of $SL(2, \mathbb{R})$

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Tempered dual of $SL(2, \mathbb{R})$

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The example of $SL(2, \mathbb{R})$
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3. $\mathbb{R}/\mathbb{Z}_2$. 

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Principle

We view $C^*\pi(G)$ as "$C_0(\hat{G}_\lambda)$", and $C(G)$ as "$\mathcal{S}(\hat{G}_\lambda)$", and $K^*(C^*\pi(G)) = K^*(C(G)) = K^*(\hat{G}_\lambda)$. 

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At the algebra level, we have

$$C^*_r(SL(2, \mathbb{R})) \sim \bigoplus_{n \neq 0} \mathbb{C} \oplus C_0(\mathbb{R}) \rtimes \mathbb{Z}_2 \oplus C_0(\mathbb{R}/\mathbb{Z}_2).$$
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Principle

We view $C^*_r(G)$ as “$C_0(\hat{G}_\lambda)$”, and $\mathcal{C}(G)$ as “$S(\hat{G}_\lambda)$”, and $K_\bullet(C^*_r(G)) = K_\bullet(\mathcal{C}(G))$ as $K^\bullet(\hat{G}_\lambda)$.
Decomposition of $C^*_r(G)$

The structure of $C^*_r(G)$ is studied by Wassermann and Clare-Crisp-Higson. It is shown that $C^*_r(G)$ and also $C(G)$ have the following decomposition,

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For each pair $[P,\sigma]$, there is a connected abelian Lie group $A_P$ together with a finite group $W_\sigma$ of the form $W'_\sigma \rtimes R_\sigma$ that acts faithfully on $\hat{A}_P$. 

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For each pair $[P,\sigma]$, there is a connected abelian Lie group $A_P$ together with a finite group $W_{\sigma}$ of the form $W'_{\sigma} \rtimes R_{\sigma}$ that acts faithfully on $\hat{A}_P$.

The component $C^*_r(G)[P,\sigma]$ is Morita equivalent to

$$C_0(\hat{A}_P/W'_{\sigma}) \rtimes R_{\sigma}.$$
Essential components

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The $K$-theory group of these components can be computed,

$$K_0(C_0(\hat{A}_P)) = \mathbb{Z}, \quad K_0(C_0(\hat{A}_P) \rtimes R_\sigma) = \mathbb{Z}.$$
Let us look at Poisson geometry on the essential components.
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- When $W_\sigma = R_\sigma = 1$, we have usual Poisson structures on $\hat{A}_P$. 

**Theorem (Pflaum-Posthuma-T)**

$$\text{HH}^\bullet(\mathcal{C}_\infty^c(\hat{A}_P) \rtimes \mathbb{R}^\sigma) = \Gamma(\wedge^\bullet T\hat{A}_P)^{\mathbb{R}_\sigma}.$$
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**Theorem (Pflaum-Posthuma-T)**

$$HH^\bullet(C_c^\infty(\hat{A}_P) \rtimes R_\sigma) = \Gamma\left(\land^\bullet T\hat{A}_P\right)^{R_\sigma}.$$

We have the Dolgushev equivariant formality theorem for deformation quantization of invariant Poisson structures.

**Question**: What are the corresponding deformations of $C^*_r(G)$?
The example of $SL(2, \mathbb{R})$

Geometry of the tempered dual

Index theory

Differential currents on $\hat{G}_\lambda$

Recall that the index of $\mathcal{D}_\mu$ is an element

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In order to extract characteristic numbers for $\text{Ind}(\mathcal{D}_\mu)$, we need to consider differential currents and homology on $\hat{G}_\lambda$. The right object to work with is cyclic homology of $C(G)$. 
When $G = \mathbb{R}^n$, $\mathcal{C}(\mathbb{R}^n) = \mathcal{S}(\hat{\mathbb{R}}^n)$. 
When \( G = \mathbb{R}^n \), \( C(\mathbb{R}^n) = S(\hat{\mathbb{R}}^n) \).
The homology of \( \hat{\mathbb{R}}^n \) can be computed,

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H_{\bullet}(\hat{\mathbb{R}}^n) = \begin{cases} 
\mathbb{R}, & \bullet = n, \\
0, & \text{otherwise}.
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Differential currents on $\hat{\mathbb{R}}^n$

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$$H_\bullet(\hat{\mathbb{R}}^n) = \begin{cases} \mathbb{R}, & \bullet = n, \\ 0, & \text{otherwise.} \end{cases}$$

On $S(\hat{\mathbb{R}}^n)$, $H_n(\hat{\mathbb{R}}^n)$ is generated by a degree $n$ differential current,

$$\Psi(f_0, \cdots, f_n) = \int_{\mathbb{R}^n} f_0 df_1 \cdots df_n.$$
Cyclic cocycle on $\mathcal{C}(\mathbb{R}^n)$

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Define a function $C : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ by

$$C(x_1, \cdots, x_n) := \begin{vmatrix} x_1^1 & \cdots & x_n^1 \\ \vdots & \ddots & \vdots \\ x_1^n & \cdots & x_n^n \end{vmatrix}$$
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Define $\Phi$ to be a cocycle on $C(\mathbb{R}^n)$ by

$$\Phi(f_0, \cdots, f_n) := \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} dx_1 \cdots dx_n C(x_1, \cdots, x_n) f_0(-x_1 - \cdots - x_n) f_1(x_1) \cdots f_n(x_n).$$
General case

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General case

For a general connected real reductive group $G$, we can generalize the above construction.
Let $P = MAN$ be a cuspidal parabolic subgroup of $G$. Using the Iwasawa decomposition $G = KMAN$, we introduce a generalization of the determinant function

$$C : C^\infty(K \times G^{\times m}),$$

for $m = \dim(A)$. 
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for $m = \dim(A)$. For a semisimple element $x \in M$, define a degree $m$ cocycle on $\mathcal{C}(G)$ by

$$\Phi_{P,x}(f_0, f_1, \ldots, f_m) : = \int_{h \in M/Z_M(x)} \int_K \int_{G^\times m} dhdkdndg_1 \cdots dg_m$$

$$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad C(k, g_1g_2 \cdots g_m, \ldots, g_{m-1}g_m, g_m) f_0(khxh^{-1}nk^{-1}(g_1 \cdots g_m)^{-1})$$

$$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad f_1(g_1) \cdots f_m(g_m).$$
The example of \( SL(2, \mathbb{R}) \)

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Index theory

Cyclic cocycle

**Theorem (Song-T)**

The functional \( \Phi_{P,x} \) satisfies the following identities.
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The functional $\Phi_{P,x}$ satisfies the following identities.

- $\partial \Phi_{P,x} = 0$, e.g.

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\Phi_{P,x}(f_0 \ast f_1, f_2, \cdots, f_{m+1}) - \Phi_{P,x}(f_0, f_1 \ast f_2, \cdots, f_{m+1}) + \cdots + (-1)^{m+1} \Phi_{P,x}(f_{m+1} \ast f_0, \cdots, f_m) = 0.
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- $\Phi_{P,x}$ is cyclic, e.g.
  
  $$
  \Phi_{P,x}(f_m, f_0, \cdots, f_{m-1}) = (-1)^m \Phi_{P,x}(f_0, \cdots, f_m).
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Index pairing

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\langle \Phi_{P,x}, \text{Ind}(\mathcal{D}_\mu) \rangle = \sum_{w \in W_K} (-1)^w e^{w \cdot \mu(t)} \frac{\Delta^M(t)}{\Delta^M_T(t)}.
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**Theorem (Hochs-Song-T)**

Let $Y$ be $M/M \cap K$ and $W_\mu$ be the vector bundle on $Y$ associated to $\tilde{V}_\mu$, and $\text{eu}(N^x)$ be the normal bundle of $Y^x$ in $Y$.

\[
\langle \Phi_{P,x}, \text{Ind}(\mathcal{D}_\mu) \rangle = \int_{Y^x} \chi_{x} \frac{\text{Td}(Y^x) \text{ch}(W_\mu)(x)}{\text{eu}(N^x)(x)}.
\]
Thank you for your attention!