Beyond Semitoric

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Global Poisson Webinar
Integrable Systems

An integrable system is a 2n-dimensional symplectic manifold $(M, \omega)$, and a function $F = (f_1, \ldots, f_n): M \to \mathbb{R}^n$ so that
- $\{f_i, f_j\} = 0$ for all $i, j$, and
- $F$ is regular on a dense set.

Example

- $M = \mathbb{C}$ and $F(z) = \Re(z)$. [Regular]
- $M = \mathbb{C}$ and $F(z) = |z|^2$. [Elliptic]
- $M = \mathbb{C}$ and $F(z) = \Re(z^2)$. [Hyperbolic]
- $M = \mathbb{C}^2$ and $F(x, y) = (|x|^2 - |y|^2, \Re(xy))$. [Focus-Focus]
- Any product of the first two examples. [Toric]

Here, $\omega \in \Omega^2(\mathbb{C})$ is $\sqrt{-1}dz \wedge d\overline{z}$, and $\Re(x + \sqrt{-1}y) = y$. 
Semitoric Systems

A 4-dimensional integrable system \((M, \omega, F = (\Phi, g))\) is **semitoric** if

- \(\Phi\) generates an \(S^1\) action, and
- every point either
  - has toric type, or
  - has focus-focus type.

Here, two points with isomorphic neighborhoods have same **type**.

Note: This agrees with the usual definition by work of Eliasson, Vu Ngoc & Waceux, Chaperon, and Miranda & Zung.

An integrable systems \((M, \omega, F)\) is **toric** if \(F\) generates an \((S^1)^n\) action.

Note: In this case, every point has toric type.
Semitoric Systems: Results

Theorem (Vu Ngoc)

If $(M, \omega, F)$ is compact and semitoric, then $F^{-1}(\eta, c)$ is connected for all $\eta \in \mathbb{R}$, $c \in \mathbb{R}$.

Further results

- Complete Classification [Pelayo & Vu Ngoc; Palmer, Paleyo, & Tang]
- Minimal models [Kane, Palmer & Peleyo]
- Progress on Quantization [Le Floch, Peleyo & Vu Ngoc]

Pros:

- Semitoric systems are well understood, and
- there are many interesting examples.

Cons:

- We want more examples.
Example

Let \( M = S^2 \times S^2 \) with the product symplectic form. Let \( \Phi \) be the moment map for rotating the first component. \( M \) has two fixed spheres.

Claim: \( M \) is (secretly) toric.
Examples: Semitoric but not toric

Example

Let $\hat{M}$ be the blowup of $M$ at three points in one fixed sphere. $\hat{M}$ has two fixed spheres and three fixed points $p_1$, $p_2$, $p_3$. If we blow up by the same amount, $\hat{\Phi}(p_1) = \hat{\Phi}(p_2) = \hat{\Phi}(p_3)$.

Claim: $\hat{M}$ is not toric [Karshon]. But it is semitoric [Hohloch, Sabatini, Sepe].
Example

Let $\tilde{M}$ be the blowup of $\hat{M}$ at $p_1$, $p_2$, and $p_3$ by the same amount. $\tilde{M}$ has two fixed spheres six fixed points, and three spheres fixed by $\mathbb{Z}_2$ with the same moment image.

Claim: $\tilde{M}$ is not semitoric [Hohloch, Sabatini, Sepe, Symington].
Complexity one spaces

A complexity one space is

- a $2n$-dimensional symplectic manifold $(M, \omega)$, and
- an effective $T := (S^1)^{n-1}$ action with moment map $\Phi : M \to \mathbb{R}^{n-1}$.

Example

Identify $H \subseteq T$ with a subgroup of SU$(h + 1)$, where $h = \dim H$.
Let $\Phi_H : \mathbb{C}^{h+1} \to \mathfrak{h}^*$ be the moment map for the $H$ action on $\mathbb{C}^{h+1}$.
Define a local model $Y = T \times_H \mathfrak{h}^o \times \mathbb{C}^{h+1}$.
There's an invariant $\omega \in \Omega^2(Y)$ with moment map $\Phi([t, \eta, z]) = \eta + \Phi_H(z)$.

$Y$ is tall if $Y \sslash T := \Phi^{-1}(0)/T$ contains more than one point.

Claim: Every orbit has a neighborhood isomorphic to a neighborhood of $[t, 0, 0]$ in some $Y$. 
Defining polynomials

**Lemma (Karshon-T)**

If $Y$ is tall, there exists $\xi \in \mathbb{Z}_{\geq 0}^{h+1}$ so that the defining polynomial

$$P([t, \eta, z]) = \prod z_i^{\xi_i}$$

induces a homeomorphism from $Y//T$ to $\mathbb{C}$; $P$ has degree $N := \sum_i \xi_i$

**Example**

- $Y = \mathbb{C}^2$, $\lambda \cdot (x, y) = (\lambda^px, \lambda^{-q}y)$ for $p, q > 0$, $\Phi(x, y) = p|x|^2 - q|y|^2$, $P(x, y) = x^qy^p$.  
- $Y = \mathbb{C}^3$, $(\alpha, \beta) \cdot (x, y, z) = (\alpha\beta x, \alpha^{-1}y, \beta^{-1}z)$, $\Phi(x, y, z) = (|x|^2 - |y|^2, |x|^2 - |z|^2)$, $P(x, y, z) = xyz$.  
- $Y = S^1 \times_{\mathbb{Z}_2} \mathbb{R} \times \mathbb{C}$, $\lambda \cdot [\alpha, \eta, z] = [\lambda\alpha, \eta, z]$, $\Phi([\lambda, \eta, z]) = \eta$, $P([\lambda, \eta, z]) = z^2$.  

Ephemeral critical points

Let \( g: Y \to \mathbb{R} \) be \( T \) invariant, and let \( p = [1, 0, 0] \).

Given \( \ell \geq 0 \), let \( T^\ell_p g \) be the degree \( \ell \) Taylor polynomial of \( g \) at \( p \).

This induces a function \( T^\ell_p g: Y \parallel T \to \mathbb{R} \).

\( p \) is an ephemeral critical point of \( g \) if

- \( N > 1 \),
- \( T^{N-1}_p g = 0 \), and
- The zero set of \( T^N_p g \) is homeomorphic to \( \mathbb{R} \).

Example

- \( Y = \mathbb{C}^2, \Phi(x, y) = |x|^2 - |y|^2, g(x, y) = \Im(xy) \).
- \( Y = \mathbb{C}^2, \Phi(x, y) = p|x|^2 - q|y|^2, g(x, y) = \Im(x^qy^p) \).
- \( Y = \mathbb{C}^3, \Phi(x, y, z) = (|x|^2 - |y|^2, |x|^2 - |z|^2), g(x, y, z) = \Im(xyz) \).
- \( Y = S^1 \times_{\mathbb{Z}_2} \mathbb{R} \times \mathbb{C}, \Phi([\lambda, \eta, z]) = \eta, g([\lambda, \eta, z]) = \Im(z^2) \).
Near Toric

A completely integrable system \((M, \omega, F = (\Phi, g))\) is **near toric** if

- \(\Phi\) is the moment map of a complexity one \(T\) action, and
- every point either
  - has toric type, or
  - is an ephemeral critical point of \(g\).

**Example**

Every semitoric system is near toric.

**Theorem (Sepe-T)**

*If \((M, \omega, F)\) is compact and near toric, then \(F^{-1}(\eta, c)\) is connected for all \(\eta \in \mathfrak{t}^*, c \in \mathbb{R}\).*
Proof

“Proof”.

WLOG $M//\left(S^1\right)^{n-1} := \Phi^{-1}(\eta)/(S^1)^{n-1}$ contains more than one point. So it’s a closed, oriented surface $\Sigma$ with induced function $\bar{g} : \Sigma \rightarrow \mathbb{R}$. There’s a smooth structure on $\Sigma$ so that $\bar{g}$ is a Morse function. Orbits of toric type become regular points or critical points of index 0 or 2. Ephemeral critical orbits become regular points. Since $\bar{g}$ has no points of index 1, $\bar{g}^{-1}(c)$ is connected. Hence, $F^{-1}(\eta, c) = \Phi^{-1}(\eta) \cap g^{-1}(c)$ is connected.

Bonus “proof”: $\hat{M}$ isn’t toric and $\tilde{M}$ isn’t semitoric.

Assume that $\eta$ is in the interior of $\Phi(M)$. Since $\bar{g}$ has no points of index 1, it has two critical points. Therefore, $\Phi^{-1}(\eta)$ has at most two orbits of toric type that aren’t free.
Claim: Let \((M, \omega, F = (\Phi, g))\) be an integrable system such that
- \(\Phi\) is the moment map of a complexity one \(T\) action, and
- every critical point of \(F\) is non-degenerate with no hyperbolic blocks.
Then \((M, \omega, F)\) is near toric.

Note: Wacheux studied these integrable systems.

Claim: There’s a coordinate free definition of “ephemeral” using jets which is easier to check and shows that it doesn’t depend on

Claim: Our main theorem holds whenever \(\Phi\) is proper.

Question: Is every complexity one space of genus 0 a near toric system?