Multisymplectic (co-)momentum geometry

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University of Duisburg-Essen

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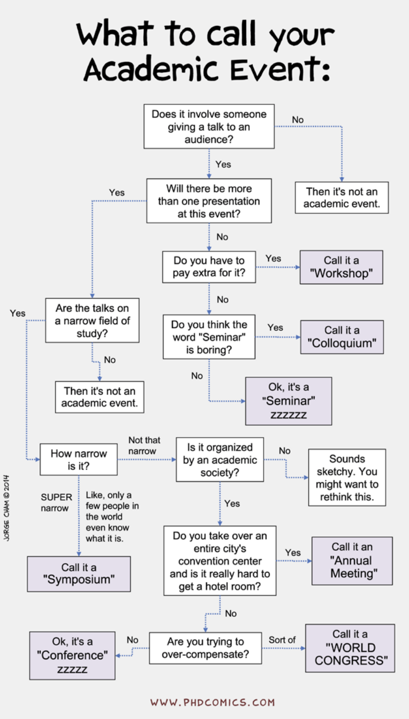
Virtual Three-Day Workshop
for Young Poisson Geometers


What to call your Academic Event:
Does it involve someone giving a talk to an audience?
No
Then it's not an academic event.
Yes
Will there be more than one presentation at this event?
No
Do you have to pay extra for it?
Yes
Call it a "Workshop"
No
Do you think the word "Seminar" is boring?
Yes
Call it a "Colloquium"
No
Ok, it's a "Seminar" zzzzzz
Yes
Are the talks on a narrow field of study?
No
Then it's not an academic event.
Yes
How narrow is it?
SUPER narrow
Like, only a few people in the world even know what it is.
Call it a "Symposium"
Not that narrow
Is it organized by an academic society?
No
Sounds sketchy. You might want to rethink this.
Yes
Do you take over an entire city's convention center and is it really hard to get a hotel room?
Yes
Call it an "Annual Meeting"
No
Are you trying to over-compensate?
Sort of
Call it a "WORLD CONGRESS"
No
Ok, it's a "Conference" zzzzz
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References

The talk is based on :

- **L.R. and T. Wurzbacher** Existence and unicity of co-moments in multisymplectic geometry, Journal of Differential Geometry and Applications, 2015.
- **A. Miti and L.R.** Multisymplectic actions of compact Lie groups on spheres, accepted to the Journal of Symplectic Geometry, 2019.
- **L. Mammadova and L.R.** On the extension problem for weak moment maps, arXiv :2001.00264, 2020.

And :

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- **Y. Fregier, C. Laurent-Gengoux, M. Zambon** A cohomological framework for homotopy moment maps, Journal of Geometry and Physics, 2015.
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Multisymplectic manifolds

Definition

Multisymplectic manifold : (M, ω), such that $\omega \in \Omega^{k+1}(M)$ is **closed** and **non-degenerate**.

Non-degeneracy : The map $\iota_\bullet \omega : TM \to \Lambda^k T^* M, \quad v \mapsto \iota_v \omega$ is injective.

Examples :

- $k = 1$: ω is symplectic.
- $k = dim(M) - 1$: ω is a volume form.
- $k = 2$: Let G be a semsisimple Lie group and $\langle \cdot, \cdot \rangle$ its Killing form. Then $\langle [\cdot, \cdot], \cdot \rangle$ extends to a biinvariant (multisymplectic) form ω.

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Reminder : Symplectic moment maps

Let (M, ω) be symplectic and $\vartheta : M \times G \to M$ a Lie group action that preserves ω.

Definition

Moment map : $f \in C^\infty(M, \mathfrak{g}^*)$, such that

1. $df^\xi(\cdot) = \omega(v_\xi, \cdot)$ for all $\xi \in \mathfrak{g}$.
2. $f(\vartheta_g(x)) = Ad_g^*(f(x))$ for all $g \in G$, $x \in M$.

Assume G is connected, then from a dual perspective,

$$f : \mathfrak{g} \to C^\infty(M)$$

1. that lifts the infinitesimal action $v : \mathfrak{g} \to \mathfrak{X}(M)$.
2. and is a Lie algebra homomorphism.

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The L_∞ algebra of observables

Let (M, ω) be multisymplectic.

$$\Omega^{k-1}_{Ham}(M,\omega) := \{\alpha \mid d\alpha = -\iota_{X_\alpha}\omega \ \text{ for some } X_\alpha \in \mathfrak{X}(M)\} \subset \Omega^{k-1}(M)$$

$$\begin{tikzcd}
L_{k-1} \arrow[r] \arrow[d, equal] & \dots \arrow[r] & \dots \arrow[r] & L_1 \arrow[r] \arrow[d, equal] & L_0 \arrow[d, equal] \\
C^\infty(M) \arrow[r] & \dots \arrow[r] & \dots \arrow[r] & \Omega^{k-2}(M) \arrow[r] & \Omega^{k-1}_{Ham}(M,\omega)
\end{tikzcd}$$

$$\{\cdots\}_i : \bigwedge^i L_0 \to L_{i-2}$$

$$\{\alpha_1, \dots, \alpha_i\}_i = -(-1)^{\frac{i(i+1)}{2}} \omega(X_{\alpha_1}, \dots, X_{\alpha_i}, \cdot, \cdots, \cdot)$$

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The L_∞ algebra of observables

Definition/ Proposition (Rogers2012)

The L_∞-algebra of multisymplectic observables : $L_\infty(M,\omega)$

1. vector spaces L_i for $i = 0, ..., k-1$
2. differential $\{\cdot\}_1 := d$
3. higher brakets $\{\cdots\}_i$ for $i = 2, ..., k+1$

is an L_∞-algebra.

$$0 = \sum_{\substack{i+j=n+1 \\ \sigma \in ush(i,n-i)}} (-1)^{i(j+1)} sgn(\sigma)\epsilon(\sigma;x)\left\{\{x_{\sigma(1)},\dots,x_{\sigma(i)}\}_i, x_{\sigma(i+1)}\cdots, x_{\sigma(n)}\right\}_j$$

Moreover, $L_\infty(M,\omega) \to \mathfrak{X}(M)$ is a L_∞-homomorphism.

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Homotopy comoments

Let (M, ω) multisymplectic and $v : \mathfrak{g} \to \mathfrak{X}(M, \omega)$ an action preserving ω.

Definition FRZ-2013

A *(homotopy) comoment* is an L_∞ homomorphism $f : \mathfrak{g} \to L_\infty(M, \omega)$ projecting to v.

$$\begin{tikzcd}
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\mathfrak{g} \arrow[ur, dashed, "f"] \arrow[r, "v"] & \mathfrak{X}(M)
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I.e. it is given by $f_i : \bigwedge^i \mathfrak{g} \to L_{i-1}$ for $i \in \{1, ..., k\}$, such that

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2. $\sum_{k<l} -(-1)^{k+l} f_{i-1}([\xi_k, \xi_l] \wedge \xi_1 ... \wedge \hat{\xi}_k \wedge ... \hat{\xi}_l \wedge ... \xi_i) = df_i(\xi_1 \wedge ... \wedge \xi_i) - (-1)^{\frac{i(i-1)}{2}} \omega(v_{\xi_1}, ..., v_{\xi_i}, \cdots)$

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Homotopy comoments

Theorem (RW15 and FLZ15)

Let (M,ω) be multisymplectic and $v : \mathfrak{g} \to \mathfrak{X}(M,\omega)$.

- The map $\xi_1 \wedge \ldots \wedge \xi_i \mapsto \pm\omega(v_{\xi_1}, \ldots, v_{\xi_i}, \cdots)$ defines classes $c_i \in H^i(\mathfrak{g}) \otimes H^{k+1-i}_{dR}(M)$
- A homotopy comoment f for the action exists if and only if $c_i = 0$ for all $i \in \{1, \ldots, k+1\}$

Symplectic case : The obstructions to the existence of a comoment live in $H^1(\mathfrak{g}) \otimes H^1_{dR}(M)$ and $H^2(\mathfrak{g}) \otimes H^0_{dR}(M)$.

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Theorem (MR19)

Let (M, ω) as above, G compact, $\vartheta : M \times G \to M$ preserves ω. A comoment exists if and only if

$$[\vartheta^*\omega - \pi^*\omega] = 0 \in H^{k+1}_{dR}(M \times G)$$

Corollaries :

- If ω has an (invariant) potential, then there is a comoment.
- If ω can be extended to an equivariant cohomology class $\tilde{\omega} \in H^{k+1}_G(M)$, then there is a comoment.
- If G_1 has a comoment on (M_1, ω_1) and G_2 has a comoment on (M_2, ω_2), then $G_1 \times G_2$ has a comoment on $(M_1 \times M_2, \pi_1^*\omega_1 + \pi_2^*\omega_2)$
- ... and on $(M_1 \times M_2, \pi_1^*\omega_1 \wedge \pi_2^*\omega_2)$. [SZ16]

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Comoments for spheres

Theorem (MR19)

Let $M = S^n$, ω a volume form and G compact, acting effectively, preserving the volume. The action admits a homotopy comoment map if and only if

- n is even,
- or the action is not transitive.

Examples :

- $SO(2n+1) \circlearrowright S^{2n}$: has a comoment
- $SO(2n) \circlearrowright S^{2n-1}$: has no comoment
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Idea of proof :

- The obstructions live in :
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- intransitive case : There exists an orbit $p \cdot G \subset S^n$ of $dim < n$.

- transitive case : classification
  - $SO(n)/SO(n-1) = S^{n-1}$
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Weak comoment maps

Let (M, ω) and G as above. $P^k_{\mathfrak{g}} = ker(\Lambda^k \mathfrak{g} \to \Lambda^{k-1}\mathfrak{g})$

Definition (H2018)

A *weak comoment map* is a collection $f_i : P^i_{\mathfrak{g}} \to L_{i-1}$ for $i \in \{1, ..., k\}$, such that

1. $df_1(\xi) = -\iota_{v_\xi}\omega$
2. $0 = df_i(\xi_1 \wedge ... \wedge \xi_i) - (-1)^{\frac{i(i-1)}{2}}\omega(v_{\xi_1}, ..., v_{\xi_i}, \cdots)$

- If a homotopy comoment exists, it exists.
- If $H^1_{dR}(M) = ... = H^k_{dR}(M) = 0$, then it exists.
- Especially, for (S^n, ω) it exists. $(n = k + 1)$.

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2. $0 = df_i(\xi_1 \wedge ... \wedge \xi_i) - (-1)^{\frac{i(i-1)}{2}}\omega(v_{\xi_1}, ..., v_{\xi_i}, \cdots)$

- If a homotopy comoment exists, it exists.
- If $H^1_{dR}(M) = ... = H^k_{dR}(M) = 0$, then it exists.
- Especially, for (S^n,ω) it exists. $(n = k+1)$.

Weak comoment maps

If $c_{k+1} \neq 0 \in H^0_{dR}(M) \otimes H^{k+1}(\mathfrak{g})$, then there is no homotopy comoment, but there still might be a weak comoment.

Theorem (MR2020)

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Thank you for your attention !