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POISSON AND QUASI-POISSON STRUCTURES FOR NONABELIAN INTEGRABLE SYSTEMS

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#JGPW2020

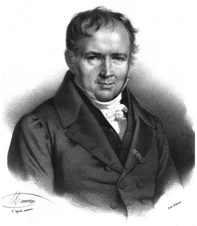
Plenary session – 15 Sep 2020

Athena SWAN Silver Award
ECU GENDER CHARTER

Outline

What is a Poisson bracket?

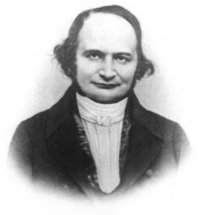
We all know where this story was born



S.D. Poisson (1809)

$$f_i(t, q_1, q_2, \dots, q_n, p_1, \dots, p_n) = a_i \text{ (const.)}$$

$$(f_j, f_k) = \sum_{l=1}^{n} \left(\frac{\partial f_j}{\partial q_l} \frac{\partial f_k}{\partial p_l} - \frac{\partial f_j}{\partial p_l} \frac{\partial f_k}{\partial q_l} \right) = \text{const.}$$



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$$(A, (B, C)) + (B, (C, A)) + (C, (A, B)) = 0$$

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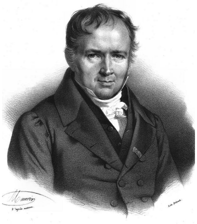

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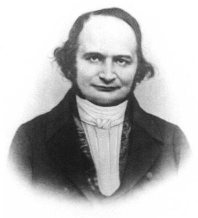

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What is a Poisson bracket?

Definition (Poisson bracket)

A *Poisson algebra* over a (commutative) ring (with unit) $\mathbb{K}$ is a triple $(A,\cdot,\{-,-\})$ where $(A,\cdot)$ is an associative $\mathbb{K}$-algebra and $(A,\{-,-\})$ is a $\mathbb{K}$-Lie algebra, such that

$$\{a, b\cdot c\} = \{a,b\}\cdot c + b\cdot\{a,c\}$$

for any $a,b,c\in A$.

1. $\{\alpha a+\beta b, c\} = \alpha\{a,c\}+\beta\{b,c\}$ (bi)linearity
2. $\{a,b\} = -\{b,a\}$ skewsymmetry
3. $\{a,\{b,c\}\} = \{\{a,b\},c\}+\{b,\{a,c\}\}$ Jacobi identity
4. $\{a,bc\} = \{a,b\}c + b\{a,c\}$ Leibniz property

Jargon A Lie bracket which is also a derivation

What *really* is a Poisson bracket?

A (mechanical, physical, ...) system: (configuration)/phase/fields space P
A way to extract information from the system: "observables", $f \in C^\infty(P)$

1. A Lie algebra for the observables
2. An (infinitesimal) action of $C^\infty(P)$ on P (by derivations!, X_h)

such that

$$X_{\{f,g\}} = -[X_f, X_g]$$

What *really* is a Poisson bracket?

A (mechanical, physical, ...) system: (configuration)/phase/fields space $\mathcal{A}$
A way to extract information from the system: "observables", $f \in \mathcal{F}$

1. A Lie algebra for the observables
2. An (infinitesimal) action of $\mathcal{F}$ on $\mathcal{A}$ (by derivations!, X_h)

such that

$$X_{\{f,g\}} = -[X_f, X_g]$$

TWO roles for ONE bracket!

Poisson brackets in Integrable Systems

- Finite dimensional systems (systems of ODEs): classical picture
- Systems of PDEs: formal calculus of variations
  - Poisson structure is given by (pseudo)differential operators
  - Modern theory of Integrable Systems: Zakharov, Shabat, Magri,. . . ,Dubrovin,. . .
  - Algebraic approach: Poisson vertex algebras (De Sole, Kac and collabs.
- Systems of differential-difference equations: technical difficulties but similar to PDEs (Kuperschmidth 1980)

Outline

The path to nonabelian systems

$$\begin{cases} \dot{u} = u^2 v - v u^2 \\ \dot{v} = 0 \end{cases}$$

is integrable for $u, v \in \mathfrak{gl}(n)$, $\forall n$.

- What for?
  1. Rigid body: $u \in \mathfrak{so}(n)$, $v = \mathrm{diag}(a_1, \dots, a_n)$ (Manakov)
  2. Nonabelian (periodic) Volterra chain: $u, v \in \mathfrak{gl}(n \cdot m)$, $u_k, v_k \in \mathfrak{gl}(m)$.

$$u = \begin{pmatrix} 0 & 0 & \cdots & u_n \\ u_1 & 0 & \cdots & 0 \\ 0 & u_2 & 0 & \vdots \\ \vdots & 0 & u_3 & \ddots \end{pmatrix} \qquad v = \begin{pmatrix} 0 & -v_1 & 0 & \cdots \\ 0 & 0 & -v_2 & 0 \\ \vdots & \vdots & 0 & -v_3 \\ -v_n & 0 & \cdots & \ddots \end{pmatrix}$$

$$(u_k)_t = v_k u_{k+1} u_k - v_{k-1} u_k u_{k-1}$$

The path to nonabelian systems

$$\begin{cases} \dot{u} = u^2 v - v u^2 \\ \dot{v} = 0 \end{cases}$$

is integrable for $u, v \in \mathfrak{gl}(n)$, $\forall n$.

- Integrable in which sense?
  Lax integrability

$$L := v\lambda - u \qquad A := \frac{u^2}{\lambda}$$

$$L_t = [L, A] \qquad \mathtt{tr} L^k \text{ are conserved quantitites}$$

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And then we forget about the original matrix formulation and deal only with non-commuting functions $u(t), v(t)$.

- PDE case: $u_t = u_{3x} + u u_x + u_x u$ (nonabelian KdV)
- DΔE case: $u_t = u_1 u - u u_{-1}$ (nonabelian (open) Volterra)

But what does it mean to be a Hamiltonian system?

Operational	Algebraic	Geometric
Olver & Sokolov	Van Den Bergh De Sole, Kac & Valeri	$\sim$ Ping Xu Kiselev

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The setting

We keep the crucial difference between $\mathcal{A}$ and $\mathcal{F}$ (and yes, now it applies to ODEs too)

	$\mathcal{A}$	$\mathcal{F}$
ODEs	Noncomm. Laurent polyn.	$\dfrac{\mathcal{A}}{[\mathcal{A},\mathcal{A}]}$
	$\mathcal{A}=\dfrac{\mathbb{K}\langle\{u^i,(u^i)^{-1}\}_{i=1,\dots,\ell}\rangle}{\langle u^j(u^j)^{-1}-1\rangle}$	$f\in\mathcal{A}$, $\operatorname{tr} f\in\mathcal{F}$
PDEs	Noncomm. L. differential p.	$\dfrac{\mathcal{A}}{[\mathcal{A},\mathcal{A}]+\partial\mathcal{A}}$
	$\mathcal{A}=\dfrac{\mathbb{K}\langle\{u^i_{nx},\left(u^i_{nx}\right)^{-1}\}_{i=1,\dots,\ell,n\geq 0}\rangle}{\langle u^j_{kx}(u^j_{kx})^{-1}-1\rangle}$	$\int\operatorname{tr} f\in\mathcal{F}$
DΔE	Noncomm. L. difference p.	$\dfrac{\mathcal{A}}{[\mathcal{A},\mathcal{A}]+(\mathcal{S}-1)\mathcal{A}}$
	$\mathcal{A}=\dfrac{\mathbb{K}\langle\{u^i_{n},\left(u^i_{n}\right)^{-1}\}_{i=1,\dots,\ell,n\in\mathbb{Z}}\rangle}{\langle u^j_{m}(u^j_{m})^{-1}-1\rangle}$	$\int\operatorname{tr} f\in\mathcal{F}$

The setting

We focus on the ODEs/“ultralocal” case for simplicity. Difference case in the parallel session.

Scalar multiplicative operators $K\colon \mathcal{A} \to \mathcal{A}$

$$\mathrm{l}_a \mathrm{r}_b f := a f b \qquad \mathrm{c}_a := \mathrm{l}_a - \mathrm{r}_a \qquad \mathrm{a}_a := \mathrm{l}_a + \mathrm{r}_a$$

$$\mathrm{l}_a \mathrm{l}_b = \mathrm{l}_{ab} \qquad \mathrm{r}_a \mathrm{r}_b = \mathrm{r}_{ba} \qquad \mathrm{l}_a \mathrm{r}_b = \mathrm{r}_b \mathrm{l}_a$$

Double derivative $\frac{\partial}{\partial u}\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$

$$\frac{\partial}{\partial u^s}\left((u^{i_1})^{\pm 1}\cdots(u^{i_n})^{\pm 1}\right) = \sum{}' \delta_{i_k,s}(u^{i_1})^{\pm 1}\cdots(u^{i_{k-1}})^{\pm 1} \otimes (u^{i_{k+1}})^{\pm 1}\cdots(u^{i_n})^{\pm 1}$$
$$- \sum{}'' \delta_{i_k,s}(u^{i_1})^{\pm 1}\cdots(u^{i_{k-1}})^{\pm 1}(u^{i_k})^{-1} \otimes (u^{i_k})^{-1}(u^{i_{k+1}})^{\pm 1}\cdots(u^{i_n})^{\pm 1}$$

$$\frac{\partial}{\partial u}(ab) = \frac{\partial a}{\partial u} b + a \frac{\partial b}{\partial u}$$

The setting

Operations in $\mathcal{A}\otimes\mathcal{A}$ Let $A\in\mathcal{A}\otimes\mathcal{A}$, with Sweedler's notation, $A'\otimes A''$ $(=\sum_k A'_k\otimes A''_k)$. Let a,b in $\mathcal{A}$.

$$a(b\otimes c)d = ab\otimes cd \qquad a\star(b\otimes c)\star d = bd\otimes ac$$
$$(a\otimes b)\bullet(c\otimes d) = ac\otimes db \qquad \tau_{(12)}(a\otimes b) = b\otimes a$$
$$m(a\otimes b) = ab$$

Note that $\bullet$ product works as composition of multiplication operators $(\mathrm{l}_a\mathrm{r}_b\circ\mathrm{l}_c\mathrm{r}_d)$. We can translate between (noncommutative) multiplication operators and elements of $\mathcal{A}\otimes\mathcal{A}$ by

$$\mathrm{l}_a\mathrm{r}_b\leftrightarrow a\otimes b$$

Operations in $\mathcal{A}^{\otimes 3}$

$$(a\otimes b)\otimes_1 c = a\otimes c\otimes b \qquad \tau_{(123)}(a\otimes b\otimes c) = c\otimes a\otimes b$$

Double Poisson algebras

Van Den Bergh, 2008

Define a double bracket $\{\{-,-\}\}\colon \mathcal{A}\times\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ such that

1. $\{\{a,\beta b+\gamma c\}\}=\beta\{\{a,b\}\}+\gamma\{\{a,c\}\}$
2. $\{\{a,bc\}\}=b\{\{a,c\}\}+\{\{a,b\}\}c$ (derivation in the second entry)
3. $\{\{b,a\}\}=-\tau_{(12)}\{\{a,b\}\}$ (skewsymmetry)
4. Let $\{\{a,b\otimes c\}\}_L:=\{\{a,b\}\}\otimes c$. Then
   $$\{\{a,\{\{b,c\}\}\}\}_L+\tau_{(123)}\left(\{\{b,\{\{c,a\}\}\}\}_L\right)+\tau^2_{(123)}\left(\{\{c,\{\{a,b\}\}\}\}_L\right)=0$$
   (double Jacobi identity). We denote the expression on the RHS $\{\{a,b,c\}\}$.

Given a double bracket among the generators of $\mathcal{A}$,

$$\{\{f,g\}\}=\sum_{i,j=1\dots\ell}\frac{\partial g}{\partial u^j}\bullet\{\{u^i,u^j\}\}\bullet\tau_{(12)}\left(\frac{\partial f}{\partial u^i}\right)$$

Double Poisson algebras

The Poisson bracket

Theorem

Let $(\mathcal{A}, \{\{-,-\}\})$ *be a double Poisson algebra. Then*

1. *It defines a Lie algebra structure on* $\mathcal{F}$*, by*

$$\{F, G\} = \operatorname{tr} m(\{\{f, g\}\})$$

for $F = \operatorname{tr} f$*,* $G = \operatorname{tr} g$*.*

2. *It defines an action of* $\mathcal{F}$ *on* $\mathcal{A}$ *by derivations, by*

$$X_F(a) = m\left(X_F^i \star \frac{\partial a}{\partial u^i}\right)$$

and

$$X_F^i = -m\left(\{\{f, u^i\}\}\right)$$

3. *The two are compatible:* $X_{\{F,G\}} = -[X_F, X_G]$*.*

Double Poisson algebras

Sketch of the proof

Well-defined bracket Vanishing on commutators

Skewsymmetry of $\{-,-\}$ Let $\{\{f,g\}\} = B' \otimes B''$. Then $\{G,F\} = -\operatorname{tr} m(\{\{g,f\}\}) = +\operatorname{tr} m(\tau_{(12)}\{\{f,g\}\}) = \operatorname{tr} B''B' = \operatorname{tr} B'B'' = -\{F,G\}$.

Jacobi identity of $\{-,-\}$ We have

$$\{F,\{G,H\}\} - \{G,\{F,H\}\} - \{\{F,G\},H\} = \operatorname{tr} m\left((m\otimes 1)\{\{f,g,h\}\} - (1\otimes m)\{\{g,f,h\}\}\right) = 0$$

Double Poisson algebras

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Double Poisson algebras

Examples and Hamiltonian systems

1. Scalar case
   - $\mathrm{c}_u \leftrightarrow \{\{u, u\}\} = u \otimes 1 - 1 \otimes u$
   - $\mathrm{l}_u \mathrm{r}_{u^2} - \mathrm{l}_{u^2} \mathrm{r}_u \leftrightarrow \{\{u, u\}\} = u \otimes u^2 - u^2 \otimes u$
2. Two-components

$$\begin{cases} \dot{u} = u^2 v - v u^2 \\ \dot{v} = 0 \end{cases}$$

is Hamiltonian with respect to

$$\{\{u, u\}\} = u \otimes v - v \otimes u \qquad \{\{u, v\}\} = 0$$
$$\{\{v, u\}\} = 0 \qquad \{\{v, v\}\} = 0$$

and

$$H = \frac{1}{2} \operatorname{tr} u^2$$

Double quasi-Poisson algebras

Let A be an associative algebra whose units admits a finite decomposition in orthogonal idempotents $1 = \sum e_s$, $e_s e_t = \delta_{st} e_s$, and $\{\{-,-\}\}$ a linear double bracket (skewsymmetric, not Jacobi) satisfying

$$\begin{aligned}\{\{a,b,c\}\} = \alpha \textstyle\sum_s \big(& ce_s a \otimes e_s b \otimes e_s - ce_s a \otimes e_s \otimes be_s \\ & - ce_s \otimes ae_s b \otimes e_s + ce_s \otimes ae_s \otimes be_s \\ & - e_s a \otimes e_s b \otimes e_s c + e_s a \otimes e_s \otimes be_s c \\ & + e_s \otimes ae_s b \otimes e_s c - e_s \otimes ae_s \otimes be_s c\big)\end{aligned}$$

Then $(A, \{\{-,-\}\})$ is a *double quasi-Poisson algebra*.

Examples

- Scalar case $\{\{u,u\}\} = u^2 \otimes 1 - 1 \otimes u^2$
- Two components (Kontsevich)

$$\{\{u,u\}\} = 1 \otimes u^2 - u^2 \otimes 1 \qquad \{\{u,v\}\} = u \otimes v - v \otimes u - 1 \otimes uv - vu \otimes 1$$

$$\{\{v,u\}\} = uv \otimes 1 + 1 \otimes vu + u \otimes v - v \otimes u \qquad \{\{v,v\}\} = v^2 \otimes 1 - 1 \otimes v^2$$

Double quasi-Poisson algebras

Let $\mathcal{A}$ as above, and $\{\{-,-\}\}$ a linear double bracket (skewsymmetric, not Jacobi) satisfying

$$\begin{aligned}\{\{a,b,c\}\} = \alpha\,(&ca\otimes b\otimes 1 - ca\otimes 1\otimes b\\ &- c\otimes ab\otimes 1 + c\otimes a\otimes b\\ &- a\otimes b\otimes c + a\otimes 1\otimes bc\\ &+1\otimes ab\otimes c - 1\otimes a\otimes bc)\end{aligned}$$

Then $(\mathcal{A},\{\{-,-\}\})$ is a *double quasi-Poisson algebra*.

Examples

- Scalar case $\{\{u,u\}\} = u^2\otimes 1 - 1\otimes u^2$
- Two components (Kontsevich)

$$\{\{u,u\}\} = 1\otimes u^2 - u^2\otimes 1 \qquad \{\{u,v\}\} = u\otimes v - v\otimes u - 1\otimes uv - vu\otimes 1$$

$$\{\{v,u\}\} = uv\otimes 1 + 1\otimes vu + u\otimes v - v\otimes u \qquad \{\{v,v\}\} = v^2\otimes 1 - 1\otimes v^2$$

Double quasi-Poisson algebras

Double quasi-Poisson algebras and Poisson brackets

A double quasi-Poisson algebra defines a Poisson bracket too!

- The double bracket is skewsymmetric $\Rightarrow$ the bracket on $\mathcal{F}$ is skewsymmetric
- We have $(m \otimes 1)\{\{a, b, c\}\} = (1 \otimes m)\{\{a, b, c\}\} = 0 \Rightarrow$ the bracket on $\mathcal{F}$ satisfies the Jacobi identity
- There is the usual isomorphism between $(\mathcal{F}, \{-, -\})$ and $(\mathrm{Der}\,\mathcal{A}, [-, -])$ (proof in the geometric picture)

Poisson brackets and Poisson bivectors

The main idea

There is a rich and well-established machinery to read Hamiltonian systems with geometrical (Poisson) glasses:

- Hamiltonian operator $\iff$ Poisson bivector
- Poisson bivector $\Rightarrow$ Poisson brackets, Hamiltonian vector fields (equations of motion), Poisson cohomology (theory of deformations, integrability...)

Can we do the same (or something similar)?

1. Poly-vector fields and Schouten brackets
2. Poisson bivectors
3. Define PB, Hamiltonian vector fields, ...using only Schouten brackets

θ formalism

When differential graded manifolds enter the game

"Well"-known fact $T^*[1]M \cong \Lambda^\bullet TM$, and $T^*[1]M$ is a (odd) symplectic manifold with canonical bracket $\iff$ Schouten bracket on $\Lambda^\bullet TM$

Concrete construction Let $\{u^i\}$ be a coordinate chart on M, their canonical conjugate variables are $\{\theta_i\}$.

$$\deg u = 0 \qquad \deg \theta = 1 \qquad \deg f = |f|$$

Normally, u's are commuting variables and θ's Grassmann variables. However, in this noncommutative context neither have assigned parity under the product. Yet, derivations are graded and the trace operation is.

$$\hat{\mathcal{A}} := \mathcal{A}[\{\theta_i\}] \qquad \hat{\mathcal{F}} := \frac{\hat{\mathcal{A}}}{[\hat{\mathcal{A}}, \hat{\mathcal{A}}]_{(gc)}}$$

with $[f, g]_{(gc)} = fg - (-1)^{|f||g|} gf$, namely $\operatorname{tr} fg = (-1)^{|f||g|} \operatorname{tr} gf$

The Schouten bracket

Let us denote $\hat{\mathcal{A}}^p$ (resp. $\hat{\mathcal{F}}^p$) the homogeneous component of degree p of $\hat{\mathcal{A}}$ ($\hat{\mathcal{F}}$)

Double Schouten bracket

Let $[\![-,-]\!]\colon \hat{\mathcal{A}}^p \times \hat{\mathcal{A}}^q \to \left(\hat{\mathcal{A}} \otimes \hat{\mathcal{A}}\right)^{p+q-1}$

$$[\![f,g]\!] := \sum_{l=1}^{\ell} \frac{\partial g}{\partial u^l} \bullet \tau_{(12)}\left(\frac{\partial f}{\partial \theta_l}\right) + (-1)^{|g|}\frac{\partial g}{\partial \theta_l} \bullet \tau_{(12)}\left(\frac{\partial f}{\partial u^l}\right)$$

with $(a\otimes b)\bullet(c\otimes d) = (-1)^{(|c|+|d|)|b|} ac\otimes db$ and $\tau_{(12)}(a\otimes b) = (-1)^{|a||b|} b\otimes a$

Then

- $[\![g,f]\!] = -(-1)^{(|f|-1)(|g|-1)}\tau_{(12)}[\![f,g]\!]$
- $[\![f,[\![g,h]\!]]\!]_L = [\![[\![f,g]\!],h]\!]_L + (-1)^{(|f|-1)(|g|-1)}[\![g,[\![f,h]\!]]\!]_R$
  with $[\![a,b\otimes c]\!]_L = (-1)^{(|a|-1)|c|}[\![a,b]\!]\otimes c$, $[\![a,b\otimes c]\!]_R = b\otimes[\![a,c]\!]$, $[\![a\otimes b,c]\!]_L = [\![a,c]\!]\otimes_1 c$
- $[\![f,gh]\!] = g[\![f,h]\!] + (-1)^{(|f|-1)|h|}[\![f,g]\!]h$

The Schouten bracket

$$[-,-]\colon \hat{\mathcal{F}}^p \times \hat{\mathcal{F}}^q \to \hat{\mathcal{F}}^{p+q-1}$$

with $A = \operatorname{tr} a$, $B = \operatorname{tr} b$

$$[A, B] := \operatorname{tr} m\left([\![a, b]\!]\right)$$

Then

- $[F, G] = 0$ if $F, G \in \hat{\mathcal{F}}^0 = \mathcal{F}$
- $[X, Y]$ coincides with commutator of vector fields if $X, Y \in \hat{\mathcal{F}}^1$
- $[X, B] = \mathcal{L}_X(B)$
- $[B, A] = -(-1)^{(|A|-1)(|B|-1)}[A, B]$
- $[A, [B, C]] = [[A, B], C] + (-1)^{(|A|-1)(|B|-1)}[B, [A, C]]$

From the bivector to the bracket

Operators and bivectors

We say that a bivector P is *Poisson* if

$$[P, P] = 0$$

Given a multiplicative operator $K\colon \mathcal{A}^\ell \to \mathcal{A}^\ell$ we associate the bivector

$$P = \sum_{i,j=1\dots,\ell} \frac{1}{2} \operatorname{tr}\left(\theta_i K^{ij}(\theta_j)\right)$$

Example $K = \mathrm{l}_u \mathrm{r}_{u^2} - \mathrm{l}_{u^2} \mathrm{r}_u$, $P = \operatorname{tr}\left(\theta u \theta u^2\right)$

Theorem

$$[P, P] = 0 \iff \{\{-, -, -\}\} = 0$$

for $\{\{u^i, u^j\}\} = K_L^{ji} \otimes K_R^{ji}$.

From the bivector to the bracket

The Poisson bracket

Once we have the Schouten bracket and a reasonable definition of Poisson bivector, we are done: for F,G in $\mathcal{F}$

$$\{F, G\} := [[P, F], G]$$

- Skewsymmetry: from graded skewsymmetry of $[-, -]$
- Jacobi identity: $\{F, \{G, H\}\} + \circlearrowleft \iff [[[[P, P], F], G], H]$ from graded Jacobi of $[-, -]$. P Poisson $\Rightarrow$ Jacobi identity
- Action on $\mathcal{A}$: the Hamiltonian vector field for H is $[P, H] = -\,\mathtt{tr}(X_H)^i\theta_i$. Then the action on $\mathcal{A}$ is

$$X_H(f) = m\left(\llbracket X_H, f\rrbracket\right)$$

Outline

(Not a) conclusion

Hamiltonian structures beyond Poisson ones?

We have seen the case of double quasi-Poisson algebras: $[P, P] \neq 0$ but $\{F, G\} = [[P, F], G]$ a Poisson bracket.
Indeed,

$$\text{qPoisson} \iff [[P, P], F] = 0 \quad \forall F \in \mathcal{F}$$

But what we actually need is

$$\text{Jacobi} \iff [[[[P, P], F], G, H] = 0 \quad \forall F, G, H$$

$$\text{Action on } \mathcal{A} \iff [[[P, P], F], G] = 0 \quad \forall F, G$$

Are there more general structures? And integrable systems with them?

In the parallel session...

According to the interest

- Nonabelian integrable difference systems
  1. Difference operators and *multiplicative double PVA*
  2. Examples
- Poisson “geometry”
  1. Poly-vector fields and θ formalism
  2. Poisson cohomology

Thanks for your attention!

Finite-dimensional case

Classical stuff. But also, the birthplace of Poisson geometry

Hamiltonian mechanics

The phase space P with $\dim P = 2n$, coordinates (q^i, p_i), $i = 1, \ldots, n$.
The Hamiltonian function $H(q, p)$

$$\begin{cases} \dot{q}^i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q^i} \end{cases} \qquad X_h = \begin{pmatrix} \mathbf{0} & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0} \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}$$

$$\{f, g\} = \sum_{i,j=1}^{n} \frac{\partial f}{\partial z^i} \{z^i, z^j\} \frac{\partial g}{\partial z^j}$$

with

$$\{q^i, q^j\} = \{p_i, p_j\} = 0 \qquad \{q^i, p_j\} = \delta^i_j$$

Finite-dimensional case

Classical stuff. But also, the birthplace of Poisson geometry

Let F_1 such that $\{F_1, H\} = 0$. We call F_1 a first integral of the system: it is a constant of the motion and its associated Hamiltonian vector field is a symmetry of the system

$$[X_h, X_{F_1}] = 0$$

Theorem (Jacobi)

Let F_1, F_2 be first integrals of the Hamiltonian system X_h. Then $\{F_1, F_2\}$ is a first integral too

Definition (Liouville integrability)

Let $F_0 = h$, $F_1 \ldots$, F_{n-1} functionally independent first integrals of X_h, such that

$$\{F_i, F_j\} = 0 \qquad \forall i, j.$$

Then we call the system X_h integrable.

Finite-dimensional case

Classical stuff. But also, the birthplace of Poisson geometry

Let M be a smooth manifold, $\dim M = 2n + r$, and let $\Pi \in \Gamma(M, \Lambda^2 TM)$ such that

$$[\Pi, \Pi] = 0$$

(where $[-, -]$ is the Schouten bracket $\Lambda^p TM \to \Lambda^q TM \to \Lambda^{p+q-1} TM$). Then

$$\{f, g\} := \Pi(\mathrm{d}f, \mathrm{d}g)$$
$$X_f := \Pi^{\#}\mathrm{d}f$$

(M, Π) is a Poisson manifold, $[\Pi, -] = \mathrm{d}_\Pi$ the Poisson differential, splitting theorem, ...

Hamiltonian PDEs

The modern theory of Integrable Systems

(for simplicity, just one spatial dimension)

$$u(x,t)\colon \Sigma \times \mathbb{R} \to M \qquad\qquad u_x := \partial_x u\,(x,t)$$

$$\frac{\partial u^i}{\partial t} = F^i(u, u_x, u_{2x}, \dots)$$

Space of fields $\mathcal{A}$: differential polynomials/functions, densities of local functionals

$$f \in \mathcal{A} = C^\infty(\{u^i\})[\{u^i_{kx}\}]$$

with $k > 0$ and $i = 1, \dots, \ell$ (# of components, $\dim M$)
Local functionals $F = \int f$

$$F \in \mathcal{F} = \frac{\mathcal{A}}{\partial \mathcal{A}}$$

Hamiltonian PDEs

The modern theory of Integrable Systems

(for simplicity, just one spatial dimension)

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$$F \in \mathcal{F} = \frac{\mathcal{A}}{\partial \mathcal{A}}$$

Hamiltonian PDEs

The modern theory of Integrable Systems

(Proto)typical example: KdV

$$u_t = uu_x + u_{3x}$$

is (bi)Hamiltonian, in the sense

$$u_t = P_1\left(\frac{\delta H_1}{\delta u}\right) = P_2\left(\frac{\delta H_0}{\delta u}\right)$$

for

$$H_1 = \frac{1}{2}\int\left(\frac{u^3}{3} - u_x^2\right) \qquad H_0 = \frac{1}{2}\int u^2$$

$$P_1 = \partial \qquad P_2 = \frac{1}{3}\left(2u\partial + u_x\right) + \partial^3$$

Hamiltonian PDEs

$\{H_0, H_1\}_{1,2} = 0$ for $\{-,-\}_{1,2} : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$

$$\{F, G\}_{1,2} := \int \left[\frac{\delta F}{\delta u} P_{1,2} \left(\frac{\delta G}{\delta u} \right) \right]$$

The bracket (called by the community "Poisson bracket")

- is a Lie algebra bracket
- cannot be a derivation on $\mathcal{F}$ (no product there)
- defines an action of $\mathcal{F}$ on $\mathcal{A}$ by derivations: $\left(\partial^k X_H\right) \frac{\partial}{\partial u_{kx}}$ for $X_H = P\left(\frac{\delta H}{\delta u}\right) = u_t$
- $X_{\{F,G\}} = -[X_F, X_G]$

Differential-difference systems

For a discrete space

(for simplicity, just one spatial dimension)

$$u(n,t)\colon \Gamma \subseteq \mathbb{Z}\times\mathbb{R} \to M \qquad \mathcal{S}u(n,t) = u(n+1,t) =: u_1$$

$$\frac{\partial u^i}{\partial t} = F^i(u, u_1, u_{-1}, \dots)$$

Space of fields $\mathcal{A}$: difference polynomials/functions, densities of local functionals

$$f \in \mathcal{A} = \mathbb{K}[\{u^i_n\}]$$

with $n \in \mathbb{Z}$ and $i = 1, \dots, \ell$ (# of components, $\dim M$)
Local functionals $F = \int f$ (in fact, $\sum_n f(u(n))$)

$$F \in \mathcal{F} = \frac{\mathcal{A}}{(\mathcal{S}-1)\mathcal{A}} \qquad \left(\int \mathcal{S}f = \int f\right)$$

Differential-difference systems

For a discrete space

(Proto)typical example: Volterra chain

$$u_t = uu_1 - uu_{-1}$$

is (bi)Hamiltonian, in the sense

$$u_t = K_1\left(\frac{\delta H_1}{\delta u}\right) = K_2\left(\frac{\delta H_0}{\delta u}\right)$$

for

$$\frac{\delta F}{\delta u} = \sum_n \mathcal{S}^{-n}\frac{\partial f}{\partial u_n} \qquad H_1 = \frac{1}{2}\int u \qquad H_0 = \frac{1}{2}\int \log u$$

$$K_1 = u\left(\mathcal{S} - \mathcal{S}^{-1}\right)u \qquad K_2 = u\left(\mathcal{S}u\mathcal{S} + u\mathcal{S} + \mathcal{S}u - u\mathcal{S}^{-1} - \mathcal{S}^{-1}u - \mathcal{S}^{-1}u\mathcal{S}^{-1}\right)u$$

Differential-difference systems

For a discrete space

$\{H_0, H_1\}_{1,2} = 0$ for $\{-,-\}_{1,2} \colon \mathcal{F} \times \mathcal{F} \to \mathcal{F}$

$$\{F, G\}_{1,2} := \int \left[\frac{\delta F}{\delta u} K_{1,2} \left(\frac{\delta G}{\delta u} \right) \right]$$

This “Poisson bracket”, again,

- is a Lie algebra bracket
- cannot be a derivation on $\mathcal{F}$ (no product there)
- defines an action of $\mathcal{F}$ on $\mathcal{A}$ by derivations: $\left(\mathcal{S}^k X_H \right) \frac{\partial}{\partial u_n}$ for $X_H = K \left(\frac{\delta H}{\delta u} \right) = u_t$
- $X_{\{F,G\}} = -[X_F, X_G]$