Noncommutative Poisson geometry and pre-Calabi-Yau algebras

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1. Recap

Set up: $(A, \partial_A) := \left(\bigoplus_{n\in\mathbb{Z}} A^n, \partial_A\right)$: a unital dg associative $\Bbbk$-algebra (maybe, locally finite dimensional and with 1).

Def [Van den Bergh'08]
Let $d \in \mathbb{Z}$. A double Poisson bracket (resp. a double quasi-Poisson bracket) on A of degree $-d$ is a closed morphism of dg vector spaces $\{\!\{-,-\}\!\}: A[d] \otimes A[d] \longrightarrow A \otimes A$ of degree d, satisfying that

(i) Antisymmetry: $\{\!\{a,b\}\!\} = -(-1)^{(|a|-d)(|b|-d)}\sigma_{(12)}\{\!\{b,a\}\!\}$;
(ii) Leibniz identity: $\{\!\{c,ab\}\!\} = \{\!\{c,a\}\!\}b + (-1)^{(|c|-d)|a|}a\{\!\{c,b\}\!\}$;
(iii) Double Jacobi identity: $\{\!\{c,\{\!\{b,a\}\!\}\}\!\}_L + (-1)^{(|c|+d)(|a|+|b|)}\sigma_{(123)}\{\!\{b,\{\!\{a,c\}\!\}\}\!\}_L + (-1)^{(|a|+d)(|b|+|c|)}\sigma^2_{(123)}\{\!\{a,\{\!\{c,b\}\!\}\}\!\}_L = 0$.
$\left(\text{resp.} = \frac{1}{12}\left(3\left(ac\otimes b\otimes 1 - ac\otimes 1\otimes b - a\otimes cb\otimes 1 + a\otimes c\otimes b + c\otimes 1\otimes ba - c\otimes b\otimes a + 1\otimes cb\otimes a - 1\otimes c\otimes ba\right)\right)\right)$.

Def
Given $d \in \mathbb{Z}$, an A_∞-algebra B is called d-cyclic if it carries a nondegenerate bilinear form $\gamma: B\otimes B \to \Bbbk$ of degree d such that $\gamma(b_1,b_2) = (-1)^{|b_1||b_2|}\gamma(b_2,b_1)$, and $\gamma\left(m_n(b_1,\dots,b_{n-1},b_n),b_0\right) = (-1)^{n+|b_0|(\sum_{i=1}^n |b_i|)}\gamma\left(m_n(b_0,b_1,\dots,b_{n-1}),b_n\right)$ .If we drop the non degeneracy assumption on γ, A is called a degenerate d-cyclic A_∞-algebra.

Let $\partial_{d-1}A := A \oplus A^*[d-1]$. We define the natural bilinear form $\Gamma: (\partial_d A)^{\otimes 2} \to \Bbbk$ by

$$\Gamma(tf,a) = (-1)^{|a||tf|}\Gamma(a,tf) = f(a), \quad \text{and} \quad \Gamma(a,b) = \Gamma(tf,tg) = 0.$$

Def [Kontsevich—Vlassopoulos]
Given $d \in \mathbb{Z}$, a d-pre-Calabi-Yau algebra on A is the datum of a $(d-1)$-cyclic A_∞-algebra on $\partial_{d-1}A$ for the natural bilinear form Γ (of degree $d-1$) such that $m_n(A^{\otimes n}) \subset A$, for all $n \in \mathbb{N}$. A 0-pre–Calabi-Yau algebra is a pre-Calabi-Yau algebra.

Theorem 1 [Fernández—Herscovich'19]

The map

$$\left\{ \begin{array}{c} \text{nice } d\text{-pre-Calabi-Yau} \\ \text{structures } \{m_i\}_{i=1,2,3} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{double Poisson brackets} \\ \text{on } A \text{ of degree } -d \end{array} \right\}$$

given by sending m_3 to the double Poisson bracket determined by

$$(f \otimes g)\big(\{\!\{a, b\}\!\}\big) = s^{a,b}_{f,g}\, \Gamma\big(m_3(b, tg, a), tf\big)$$

is a bijection.

Theorem 2 [Fernández—Herscovich'20]

Let A be an associative $\Bbbk$-algebra carrying a double quasi-Poisson bracket $\{\!\{ -, - \}\!\}$. Then, $\partial_{-1}A$ provided with the usual multiplication m_2, as well as the maps m_3 and $\{m_n \mid n \in 2.\mathbb{N}_{\geq 2}\}$ explicitly defined before, is a strictly unitary A_∞-algebra, defining a structure of a pre-Calabi-Yau algebra on A.

2. Pre-Calabi-Yau algebras and noncommutative P_∞-algebras

Starting Point: Roughly speaking, a P_∞-algebra [Cattaneo—Felder'07] is a graded commutative algebra with an L_∞-structure such that the multibrackets satisfy a Leibniz rule. It can be regarded as a generalization of a Poisson structure, where the Jacobi identity holds only "up to homotopy".

[Schedler'09]: Define "infinity" version of double Poisson algebras. The latter arise by considering the double Jacobi identity up to homotopies, but not the associativity of the multiplication.

Def [Schedler'09; Fernández—Herscovich'19]

A double P_∞-algebra is a graded algebra $A = \oplus_{n\in\mathbb{Z}} A^n$ provided with a family of homogeneous maps $\{\!\{\dots\}\!\}_p \colon A^{\otimes p} \to A^{\otimes p}$ indexed by $p \in \mathbb{N}$, where $\{\!\{\dots\}\!\}_p$ has degree $2-p$, such that

Antisymmetry$_\infty$: $\sigma \circ \{\!\{\dots\}\!\}_p \circ \sigma^{-1} = \mathrm{sgn}(\sigma)\{\!\{\dots\}\!\}_p$, for all $\sigma \in \mathbb{S}_n$;

Leibniz$_\infty$: For all $p \in \mathbb{N}$, and homogeneous $a_1, \dots, a_{p-1} \in A$, the map $A \to A^{\otimes p}$, $a \mapsto \{\!\{a_1, \dots, a_{p-1}, a\}\!\}_p$ satisfies

$$\{\!\{a_1, \dots, a_{p-1}, ab\}\!\}_p = \{\!\{a_1, \dots, a_{p-1}, a\}\!\}_p b + (-1)^{|a|(p+\sum_{j=1}^{p-1}|a_j|)} a\{\!\{a_1, \dots, a_{p-1}, b\}\!\}_p;$$

Jacobi$_\infty$: For all $p \in \mathbb{N}$, and $\{\!\{\dots\}\!\}_{i,p-i+1} := \left(\{\!\{\dots\}\!\}_i \otimes \mathrm{Id}_A^{\otimes(p-i)}\right) \circ \left(\mathrm{Id}_A^{\otimes(i-1)} \otimes \{\!\{\dots\}\!\}_{p-i+1}\right)$, we have

$$\sum_{i=1}^{p} (-1)^{i(p+1)} \sum_{\sigma\in C_p} \sigma \circ \{\!\{\dots\}\!\}_{,i,p-i+1} \circ \sigma^{-1} = 0.$$

Remark Note that a double P_∞-algebra with brackets $\left\{\{\!\{\dots\}\!\}_p\right\}_{p\in\mathbb{N}}$ satisfying $\{\!\{\dots\}\!\}_p = 0$ for all $p > 2$ is a double Poisson dg algebra of degree zero, with $\{\!\{-,-\}\!\}_A = \{\!\{\dots\}\!\}_2$ and $\partial_A = \{\!\{\dots\}\!\}_1$.

Natural question: Keeping in mind the link between pre-Calabi-Yau algebras and double (quasi-)Poisson algebras, Can we find a link between pre-Calabi-Yau algebras and Schedler's double P_∞-algebras?

Theorem A [Fernández—Herscovich'19]
Let $A = \oplus_{n\in\mathbb{N}} A^n$ be a graded algebra, and we consider the graded algebra structure on $\partial_{-1}A = A \oplus A^*[-1]$, provided with the natural nondegenerate bilinear form Γ of degree -1. Then,

(i) Given a pre-Calabi-Yau structure on A with multiplications $\{m_2\} \cup \{m_p\}_{p \text{ odd}}$, we define the family of maps $\{\{\!\{\dots\}\!\}_p\}_{p\in\mathbb{N}}$ with $\{\!\{\dots\}\!\}_p\colon A^{\otimes p} \to A^{\otimes p}$ given by

$$(f_1 \otimes \cdots \otimes f_p)(\{\!\{ a_1, \dots, a_p \}\!\}_p) = s^{a_1,\dots,a_p}_{f_1,\dots,f_p}\,\Gamma\Big(m_{2p-1}\big(a_p, tf_p, \dots, a_2, tf_2, a_1\big), tf_1\Big) \qquad (1)$$

for $p \in \mathbb{N}$ and all homogeneous $a_1, \dots, a_p \in A$ and $f_1, \dots, f_p \in A^*$, determines a structure of a double P_∞-algebra on the graded algebra A.

(ii) Moreover, the map

$$\left\{ \begin{array}{c} \text{essentially odd} \\ \text{pre-Calabi-Yau structures} \\ \{m_\bullet\}_{\bullet\in\mathbb{N}} \text{ on } A \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{double } P_\infty\text{-algebra} \\ \text{structures } \{\{\!\{\dots\}\!\}_\bullet\}_{\bullet\in\mathbb{N}} \text{ on } A \end{array} \right\}$$

given by sending $\{m_2\}_{\bullet\in\mathbb{N}}$ to the family of maps $\{\{\!\{\dots\}\!\}_\bullet\}_{\bullet\in\mathbb{N}}$ determined by (1) is a bijection.

Remark As in **Theorem 1**, assuming that $s^{a_1,\dots,a_p}_{f_1,\dots,f_p}$ is just of function of the degrees $|a_i|$ and $|f_i|$, we found

$$s^{a_1,\dots,a_p}_{f_1,\dots,f_p} = (-1)^{|a_p||f_1| + (p+1)(|a_p|+|f_1|) + \sum_{j=1}^{p}(p-j)|a_j| + \sum_{j=1}^{p}(j-1)|f_j|} (-1)^{\sum_{1\le i<j<p}|a_i||a_j| + \sum_{1<i<j\le p}|f_i||f_j| + \sum_{1<i\le j<p}|f_i||a_j|}.$$

Idea of the proof:

Assume that we want to prove that Leibniz$_\infty$ holds for a fixed $p \in \mathbb{N}$:

$$\{\!\{a_1, \dots, a_{p-1}, a_0 b_0\}\!\}_p = \{\!\{a_1, \dots, a_{p-1}, a_0\}\!\}_p b_0 + (-1)^{|a_0|(p+\sum_{j=1}^{p-1}|a_j|)} a_0 \{\!\{a_1, \dots, a_{p-1}, b_0\}\!\}_p, \tag{2}$$

for all homogeneous $a_0, a_1, \dots, a_{p-1}, b_0 \in A$.

We consider the Stasheff identity SI$(2p)$ for the A_∞-algebra structure of $\partial_{-1}A$, which evaluated at $a_0 \otimes b_0 \otimes tf_1 \otimes a_1 \otimes \cdots \otimes tf_{p-1} \otimes a_{p-1} \in A \otimes A \otimes A^*[-1] \otimes A \otimes \cdots \otimes A^*[-1] \otimes A$, reduces to:

$$-(-1)^{|a_0|} a_0 \,.\, m_{2p-1}(b_0, tf_1, a_1, \dots, tf_{p-1}, a_{p-1}) + m_{2p-1}(a_0 b_0, tf_1, a_1, \dots, tf_{p-1}, a_{p-1}) - m_{2p-1}(a_0, b_0 \,.\, tf_1, a_1, \dots, tf_{p-1}, a_{p-1}) = 0.$$

By applying $\Gamma(-, tf_p)$ for an arbitrary homogeneous $f_p \in A^*$,

$$-(-1)^{|a_0|}\Gamma\big(a_0 \,.\, m_{2p-1}(b_0, tf_1, a_1, \dots, tf_{p-1}, a_{p-1}), tf_p\big) + \Gamma\big(m_{2p-1}(a_0 b_0, tf_1, a_1, \dots, tf_{p-1}, a_{p-1}), tf_p\big) - \Gamma\big(m_{2p-1}(a_0, b_0 \,.\, tf_1, a_1, \dots, tf_{p-1}, a_{p-1}), tf_p\big) = 0.$$

Applying the cyclicity of Γ and the definition $(f_1 \otimes \cdots \otimes f_p)(\{\!\{a_1, \dots, a_p\}\!\}_p) = s^{a_1,\dots,a_p}_{f_1,\dots,f_p}\Gamma\Big(m_{2p-1}(a_p, tf_p, \dots, a_2, tf_2, a_1), tf_1\Big)$, we get

$$0 = -s^{a_{p-1},\dots,a_1,b_0}_{f_p,a_0,f_{p-1},\dots,f_1}(-1)^{|a_0|(p+|b_0|+|f_p|+\sum_{j=1}^{p-1}|a_j|)}(f_p \otimes \dots \otimes f_1)\big(a_0\{\!\{a_{p-1}, \dots, a_1, b_0\}\!\}_p\big)$$

$$+s^{a_{p-1},\dots,a_1,a_0b_0}_{f_p,\dots,f_1}(f_p \otimes \dots \otimes f_1)\big(\{\!\{a_{p-1}, \dots, a_1, a_0b_0\}\!\}_p\big) - s^{a_{p-1},\dots,a_0}_{f_p,\dots,f_2,b_0,f_1}(-1)^{|b_0|(p+|b_0|+|f_1|+\sum_{j=0}^{p-1}|a_j|)}(f_p \otimes \dots \otimes f_1)\big(\{\!\{a_{p-1}, \dots, a_0\}\!\}_p b_0\big).$$

Hence, (2) holds if and only if

$$s^{a_{p-1},\dots,a_1,a_0b_0}_{f_p,\dots,f_1} = s^{a_{p-1},\dots,a_0}_{f_p,\dots,f_2,b_0,f_1}(-1)^{|b_0|(p+|b_0|+|f_1|+\sum_{j=0}^{p-1}|a_j|)}, \quad \text{and} \quad s^{a_{p-1},\dots,a_1,a_0b_0}_{f_p,\dots,f_1} = (-1)^{|a_0|(|b_0|+|f_p|)} s^{a_{p-1},\dots,a_1,b_0}_{f_p,a_0,f_{p-1},\dots,f_1}.$$

Moreover, SI$(2p-1)$ for $\partial_{-1}A \iff$ Jacobi$_\infty$.

3. Double quasi-Poisson algebras are pre-Calabi-Yau (proof)

Theorem 2 [Fernández—Herscovich'20]

Let A be an associative algebra carrying a double quasi-Poisson bracket $\{\!\{-,-\}\!\}$. We consider $\partial_{-1}A = A \oplus A^*[-1]$, provided with its usual multiplication and the natural bilinear form Γ, as well as the map m_3 defined by $\Gamma\big(m_3(b, tg, a), tf\big) = (f \otimes g)(\{\!\{a, b\}\!\})$. Furthermore, for $i < j$ and $n = i + j + 1$, we define $\{m_n \colon (\partial_{-1}A)^{\otimes n} \to \partial_{-1}A \mid n \in 2.\mathbb{N}_{\geq 2}\}$ via

$$\Gamma\Big(m_n\big(tf_2, \ldots, tf_{i-1}, tf_i, a, tg_1, tg_2, \ldots, tf_{j-1}, tg_j, b\big), tf_1\Big)$$

$$= C_{i,j} \prod_{\ell=2}^{i-1} f_\ell(1) \prod_{\ell'=2}^{j-1} g_{\ell'}(1) \begin{cases} f_1(b)g_1(a)g_j(1) - f_1(ab)g_1(1)g_j(1) + f_1(a)g_1(1)g_j(b) - f_1(1)g_1(a)g_1(b) & \text{if } i = 1 \\ \Big(f_1(b)g_j(1) - f_1(1)g_j(b)\Big)\Big(f_i(1)g_1(a) - f_i(a)g_1(1)\Big) & \text{if } i \geq 2 \end{cases}$$

and $m_n = 0$ otherwise. Then, A is a strictly unitary A_∞-algebra and it defines a structure of a pre-Calabi-Yau algebra on A.

Remarks For $i, j \in \mathbb{N}$ such that $i + j$ is odd, we define

$$C_{i,j} := \binom{i+j-2}{i-1}(-1)^{\frac{i+j+1}{2}} \frac{B_{i+j-1}}{(i+j-1)!}$$

where B_ℓ is the ℓ-th Bernoulli number. For example, $C_{1,2} = \dfrac{1}{12}$, $C_{1,6} = \dfrac{2}{35}$ or $C_{1,12} = \dfrac{691}{1307674368000}$.

Idea of the proof:

By hypothesis, we assume that the modified Jacobi identity for the quasi-Poisson algebra $(A, \{\!\{ -, - \}\!\})$ holds.

Goal: We want to prove that the Stasheff identities $\mathrm{SI}(n)$ for $\partial_{-1}A = A \oplus A^*[-1]$ hold.

Instead of working with $\mathrm{SI}(n)$, we will work with the equivalent identity $(\mathrm{SI}(n)_\Gamma)$, which is defined as $\Gamma\big(\mathrm{SI}(n), -\big)$. Also, by the cyclicity of Γ, without loss of generality, we will assume that we evaluated $(\mathrm{SI}(n)_\Gamma)$ at

$$\omega = \big(a, tf_1, \ldots, tf_\ell, b, tg_1, \ldots tg_{\ell'}, c, th_1, \ldots, th_{\ell''}\big) \in A \otimes A^*[-1]^{\otimes \ell} \otimes A \otimes A^*[-1]^{\otimes \ell'} \otimes A \otimes A^*[-1]^{\otimes \ell''}$$

such that $\ell + \ell' + \ell'' = n - 2$. By **Theorem 1** and the arguments in my Plenary Talk, $(\mathrm{SI}(n)_\Gamma)$ holds when $1 \leq n \leq 5$.

To prove $(\mathrm{SI}(n)_\Gamma)$ for all integers $n > 5$, since the only nonzero higher multiplications maps are m_3 and m_n with $n \in 2.\mathbb{N}_{\geq 2}$, the only nontrivial $(\mathrm{SI}(n)_\Gamma)$ are

(A) $(\mathrm{SI}(n+2)_\Gamma)$ for all even $n \in \mathbb{N}_{\geq 4}$, which involves only m_3 and m_n;
(B) $(\mathrm{SI}(n+1)_\Gamma)$ for all even $n \in \mathbb{N}_{>4}$, which involves m_{2i}, for $i \in \mathbb{N} \cap [1, n/2]$.

The proof of (A) consists of the study of 6 cases (depending on ℓ, ℓ', ℓ'') and the use of the properties of m_3. To prove (B), we note

$$\mathrm{SI}(n+1)_\Gamma(\omega) = \mathrm{SI}(n+1)_\Gamma^{2,n}(\omega) + \mathrm{SI}(n+1)_\Gamma^{\neq 2,n}(\omega) = \sum_{k=2}^{\frac{n-2}{2}} \sum_{r=0}^{n-2k+1} (-1)^r \, \Gamma\Big(\big(m_{n-2k+2} \circ (\mathrm{id}^{\otimes r} \otimes m_{2k} \otimes \mathrm{id}^{\otimes (n-r-2k+1)})\big) \otimes \mathrm{id}\Big)(\omega)$$
$$+ \sum_{k=2}^{\frac{n-2}{2}} \sum_{r=0}^{n-2k+1} (-1)^r \, \Gamma\Big(\big(m_{n-2k+2} \circ (\mathrm{id}^{\otimes r} \otimes m_{2k} \otimes \mathrm{id}^{\otimes (n-r-2k+1)})\big) \otimes \mathrm{id}\Big)(\omega).$$

At this point,

(B1) With the non vanishing terms, we write $\mathrm{SI}(n+1)^{\neq 2,n}_{\Gamma}(\omega) = \mathrm{SI}(n+1)^{f}_{\Gamma}(\omega)+\mathrm{SI}(n+1)^{m}_{\Gamma}(\omega)+\mathrm{SI}(n+1)^{l}_{\Gamma}(\omega)$;

(B2) Plug the explicit expressions of m_n in $\mathrm{SI}(n+1)^{2,n}_{\Gamma}(\omega)$, $\mathrm{SI}(n+1)^{f}_{\Gamma}(\omega)$, $\mathrm{SI}(n+1)^{m}_{\Gamma}(\omega)$, $\mathrm{SI}(n+1)^{l}_{\Gamma}(\omega)$.

After this extremely tricky analysis of the Stasheff identity $(\mathrm{SI}(n+1)_{\Gamma})$ and comparing the obtained expressions, a lot of miraculous cancellations occur.

At the end, we obtain that $(\mathrm{SI}(n+1)_{\Gamma}) = 0$ if and only if

$$\begin{aligned}(-1)^{\ell+1}C_{\ell,\ell'+\ell''}+(-1)^{\ell''+1}C_{\ell'',\ell'+\ell}+(-1)^{\ell'+1}C_{\ell',\ell+\ell''} \\ = (-1)^{\ell'+1}\sum_{j=1}^{\lfloor \ell'/2\rfloor+\mathrm{i}_{\ell}\mathrm{i}_{\ell''}} C_{\ell,2j-\mathrm{i}_{\ell}}C_{\ell'',\ell'-2j+1+\mathrm{i}_{\ell}} + (-1)^{\ell''+1}\sum_{j=1}^{\lfloor \ell''/2\rfloor+\mathrm{i}_{\ell}\mathrm{i}_{\ell'}} C_{\ell',2j-\mathrm{i}_{\ell'}}C_{\ell,\ell''-2j+1+\mathrm{i}_{\ell'}} + (-1)^{\ell+1}\sum_{j=1}^{\lfloor \ell/2\rfloor+\mathrm{i}_{\ell'}\mathrm{i}_{\ell''}} C_{\ell'',2j-\mathrm{i}_{\ell''}}C_{\ell',\ell-2j+1+\mathrm{i}_{\ell''}},\end{aligned} \qquad \Big(\mathscr{E}q(\ell,\ell',\ell'')\Big)$$

where $\mathrm{i}\colon \mathbb{Z}\to\{0.1\}$, defined by $\mathrm{i}_j = 1$ if j is even and 0 otherwise. Fortunately, [Buijs—Carrasquel-Vera—Murillo'17] proved the following identity for the *Bernoulli numbers*, where $a,b,c\in\mathbb{N}_0$ such that $a+b+c=2k-1$:

$$-\mu_0 B_{2k} = \frac{\mu_k}{2}B_k^2 + \sum_{j=1}^{\lfloor (k-1)/2\rfloor} \mu_{2j}B_{2j}B_{2k-2j}, \qquad \Big(\mathrm{Eq}(a,b,c)\Big)$$

where

$$\mu_{2j}(a,b,c) := \binom{2k}{2j}\left[(-1)^c\binom{2k-2j}{c}\sum_{d=\max(0,2j-b)}^{\min(a,2j)}(-1)^d\binom{2j}{d} + (-1)^c\binom{2j}{c}\sum_{d=\max(0,2k-2j-b)}^{\min(a,2k-2j)}(-1)^d\binom{2k-2j}{d} - (-1)^a\binom{2k-2j}{a}\sum_{d=\max(0,2j-b)}^{\min(c,2j)}(-1)^d\binom{2j}{d} - (-1)^a\binom{2j}{a}\sum_{d=\max(0,2k-2j-b)}^{\min(c,2k-2j)}(-1)^d\binom{2k-2j}{d}\right],$$

Finally, we were able to prove $(2k+1 = \ell+\ell'+\ell''$ for an integer $k\geq 2$ and $\ell',\ell',\ell''\geq 1)$:

$$\mathscr{E}q(\ell,\ell',\ell'') = -\frac{(-1)^k}{(2k)!}\left(\frac{\ell}{2k}\mathrm{Eq}(\ell''-1,\ell'-1,\ell) + \frac{\ell'}{2k}\mathrm{Eq}(\ell''-1,\ell-1,\ell')\right).$$

4. Morphisms of pre-Calabi-Yau algebras

So far: We have focused on *objects* (i.e. double (quasi-)Poisson algebras, double P_∞-algebras, A_∞-algebras, pre-Calabi-Yau algebras ,...)

Natural question: What about morphisms?
In particular, Can we define the notion of a **morphism of pre-Calabi-Yau algebras?**

Indirect approach: Use **Theorem 1** to find morphisms of pre-Calabi-Yau algebras from morphisms of double Poisson dg algebras (which is well-known).

Def

Given $d \in \mathbb{Z}$, let $(A, \partial_A, \{\!\{-,-\}\!\}_A)$ and $(B, \partial_B, \{\!\{-,-\}\!\}_B)$ be two double Poisson dg algebras of degree d. A morphism of double Poisson dg algebras $\phi: A \to B$ is a morphism of dg algebras, such that $(\phi \otimes \phi) \circ \{\!\{-,-\}\!\}_A = \{\!\{-,-\}\!\}_B \circ (\phi[d] \otimes \phi[d])$.

Since $\phi: A \to B$ is a morphism of dg algebras, B is a dg bimodule over A, so

$$\partial_{d-1}\phi := A \oplus B^*[d-1] \,.$$

Facts:

(i) $\partial_{d-1}\phi$ carries a dg algebra structure;

(ii) $\partial_{d-1}\phi$ is naturally endowed with a super symmetric bilinear form $\Gamma_\phi: (\partial_{d-1}\phi)^{\otimes 2} \to \Bbbk$ of degree $d-1$ given by $\Gamma_\phi(tf, a) = (-1)^{|a|(|f|-d+1)}\Gamma_\phi(a, tf) = f\big(\phi(a)\big)$ and $\Gamma_\phi(a, b) = 0 = \Gamma_\phi(tf, tg)$ (for all homogeneous $a, b \in A$ and $f, g \in B^*$).

(iii) The pair $\big(\partial_{d-1}\phi\,, \Gamma_\phi\big)$ is a *degenerate* d-cyclic dg algebra.

Theorem B [Fernández—Herscovich'19]

Given $d \in \mathbb{Z}$, let $(A, \partial_A, \{\!\{ -, - \}\!\}_A)$ and $(B, \partial_B, \{\!\{ -, - \}\!\}_B)$ be two double Poisson dg algebras of degree d. Let $\phi\colon A \to B$ be a morphism of double Poisson dg algebras. By **Theorem 1**, $\big(\partial_{d-1}A, \Gamma_A, \{m_i^A\}_{i\in\{1,2,3\}}\big)$ and $\big(\partial_{d-1}B, \Gamma_B, \{m_i^B\}_{i\in\{1,2,3\}}\big)$ are endowed with pre-Calabi-Yau structures. Consider the dg algebra $\partial_{d-1}\phi = A \oplus B^*[d-1]$, and we define the unique map $m_3^\phi\colon (\partial_{d-1}\phi)^{\otimes 3} \to \partial_{d-1}\phi$ satisfying that

$$m_3^\phi(a, tf, b) = m_3^A\big(a, t(f\circ\phi), b\big), \quad \text{and} \quad m_3^\phi(tf, b, tg) = m_3^B\big(tf, \phi(b), tg\big),$$

and zero otherwise, for all homogeneous $a, b \in A$ and $f, g \in B^*$. Then, $\partial_{d-1}\phi$ is a degenerate d-cyclic A_∞-algebra, with $m_i^\phi = 0$ if $i \in \mathbb{N}\backslash\{1,2,3\}$, such that the maps

$$\Phi_A\colon \partial_{d-1}\phi \longmapsto \partial_{d-1}A, \quad (a, tf) \longmapsto \big(a, t(f\circ\phi)\big), \quad \text{and} \quad \Phi_B\colon \partial_{d-1}\phi \longmapsto \partial_{d-1}B, \quad (a, tf) \longmapsto \big(\phi(a), tf\big)$$

are strict morphisms of A_∞-algebras preserving the corresponding natural bilinear forms.

Def [Fernández—Herscovich'19]

Let A and A' be d-pre-Calabi-Yau algebras. A morphism from A to A' is a triple (C, Φ, Ψ), where C is a degenerate $(d-1)$-cyclic A_∞-algebra, and $\Phi\colon C \to A$ and $\Psi\colon C \to B$ are strict morphisms of A_∞-algebras preserving the natural bilinear forms.

Remarks:
- The key point of the proof of **Theorem B** is that the Stasheff identities SI(n) for $\partial_{d-1}\phi$ follow from SI(n) for $\partial_{d-1}A$ and $\partial_{d-1}B$, when $1 \le n \le 5$ (otherwise, the Stasheff identities are trivial);
- Moreover, we were able to define composable morphisms of pre-Calabi-Yau algebras. So, we can define a (partial) functor from the category of locally finite dimensional double Poisson dg algebras of degree d to the *partial* category of d-pre-Calabi-Yau algebras.

Thank you for your attention!