Partial compactifications of principal Poisson slices
Part II

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Joint with Markus Röser

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(iv) $\overline{X_{\text{prin}}}$ is log symplectic whenever X_{prin} is symplectic.

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Proposition

One has a canonical Poisson variety isomorphism

$$X_\tau \cong (X \times (G \times \mathcal{S}_\tau)) /\!\!/ G$$

for each Hamiltonian G-variety X.

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Proposition

There is a canonical G-equivariant symplectomorphism between $G \times \mathcal{S}_\tau$ and the unique open dense symplectic leaf in $\overline{G \times \mathcal{S}_\tau}$.

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$$X_\tau \xrightarrow{\cong} (X \times (G \times \mathcal{S}_\tau)) /\!\!/ G \hookrightarrow (X \times (\overline{G \times \mathcal{S}_\tau})) /\!\!/ G =: \overline{X_\tau}. \quad (*)$$

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Let X be a Hamiltonian G-variety, and suppose that τ is an $\mathfrak{sl}_2$-triple. We define

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- If $\overline{X_\tau}$ exists as a geometric quotient, then $(*)$ amounts to an open embedding

$$j : X_\tau \hookrightarrow \overline{X_\tau}.$$

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