Partial compactifications of principal Poisson slices
Part I

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Lie theory

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Theorem (Kostant)

If τ_1 and τ_2 are principal $\mathfrak{sl}_2$-triples, then

$$\mathcal{S}_{\tau_2} = g \cdot \mathcal{S}_{\tau_1}$$

for some $g \in G$.

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(i) One has a commutative diagram of the form

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Conclusion

One may compactify $\mathcal{Z}_{\mathfrak{g}}$ over $(T^*G)/(G \times G)$ while respecting its Poisson structure.

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Theorem (C.–Röser)

This can be achieved in the presence of certain assumptions about X.

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- Specialize to the case of a principal $\mathfrak{sl}_2$-triple τ.