Schubert calculus via Lagrangian correspondences

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Grassmannians

General setup: partial flag varieties

- G complex algebraic group, $T \subset B \subset G$, $W = N(T)/T$,
- For $B \subset P$ a parabolic, $(G/P)^T \cong W_P \backslash W \cong W/W_P$.

Study multiplication and restriction for $H_T^*(G/P)$ in a "nice" basis:

$$H_T^*(G/P) \otimes H_T^*(G/P) \to H_T^*(G/P)$$

$$H_S^*(G/P) \to H_S^*(H/Q)$$

where $H \leq G$ with compatible parabolic Q and torus S.

For G of type $A_n/B_n/C_n/D_n$, P maximal, G/P is a **Grassmannian**.

$$E.g.\ Gr(k;n) := GL_n/P_{k,n-k} \cong \{V \subseteq \mathbb{C}^n \mid \dim V = k\}$$

$$SpGr(k;2n) := Sp_{2n}/P^{Sp}_{k,2n-k} \cong \{V \subseteq \mathbb{C}^{2n} \mid \dim V = k, V \subseteq V^{\perp}\}$$

Schubert classes

Schubert classes For $\pi \in W_P\backslash W$, the corresp. **Schubert class** is

$$S_\pi := \left[\overline{B^-\pi^{-1}P/P}\right] \in H_T^*(G/P).$$

Then $\{S_\pi\}_{\pi\in W_P\backslash W}$ freely generate $H_T^*(G/P)$ as an $H_T^*(\mathrm{pt})$–module.

Classical question: Determine the structure constants,

$$S_\lambda \cdot S_\mu = \sum_\nu c_{\lambda\mu}^\nu S_\nu$$

Note: if $G/P \cong Gr(k;n)$, then (in H^*, not H_T^*) $V_\lambda \otimes V_\mu = \bigoplus_\nu V_\nu^{\oplus c_{\lambda\mu}^\nu}$

$c_{\lambda\mu}^\nu$= *the Littlewood-Richardson coefficients for* GL_k

E.g. In $Gr(2;4)$, $(H_T^*(pt) \cong \mathbb{Z}[y_1, y_2, y_3, y_4])$:

$$S_{\square} \cdot S_{\square} = S_{\square\square} + S_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} + (y_2 - y_3)S_{\square} \qquad (\text{in } H_T^*)$$

Schubert calculus via puzzles

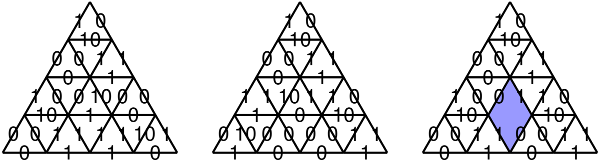
Theorem (Knutson-Tao '03, many extensions since)

For $\lambda, \mu \in 0^k 1^{n-k}$, the product of S_λ and S_μ in $H_T^*(Gr(k;n))$ is given by

$$S_\lambda \cdot S_\mu = \sum_\nu w\left(\Delta^{\nu}_{\lambda\mu}\right) S_\nu,$$

where $w\left(\Delta^{\nu}_{\lambda\mu}\right) \in H_T^*(pt)$ is computed via R-matrices in the 5–vertex model in statistical mechanics.

Example: $S_{0101} \cdot S_{0101} = S_{0110} + S_{1001} + (y_2 - y_3) S_{0101}$



Branching from A to C

We are interested in the cohomology pullback of the inclusion

$$SpGr(k;2n) \xhookrightarrow{\iota} Gr(k;2n).$$

Involution: $Sp_{2n} = GL_{2n}^{\sigma}$, for $J = \text{Antidiag}(-1,\dots,-1,1,\dots,1)$,

$$\sigma : GL_{2n} \to GL_{2n},\ X \mapsto J^{-1}(X^{-1})^{\text{tr}}J$$

Main question: $\boxed{\iota^*(S_\lambda) = \sum_\nu c_\nu^\lambda S_\nu}$ $\quad c_\nu^\lambda = ??$

- Pragacz '00: (building on work of Stembridge) positive tableau formulæ for $H^*(Gr(n;2n)) \to H^*(SpGr(n;2n))$
- Coşkun '11: positive geometric rule for $H^*(Gr(k;2n))$

Cohomology Rings

In equivariant cohomology, we get:

$$\begin{array}{ccc} H_T^*(SpGr(k;2n)^T) & \xleftarrow[f_2^*]{} & H_T^*(SpGr(k;2n)) \\ \uparrow (\tilde{\iota})^* & & \uparrow \iota^* \\ H_T^*(Gr(k;2n)^T) & \xleftarrow[f_1^*]{} & H_T^*(Gr(k;2n)) \end{array}$$

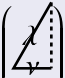
- Since each f_i^* is injective (Kirwan), to understand ι^* we can instead compute in the left column.
- Use the Andersen-Jantzen-Soergel, Billey formula ('94,'97) for restriction to T-fixed points, $S_\lambda|_\mu$.

A combinatorial branching rule

Theorem (H–Knutson–Zinn-Justin '18)

For $\lambda \in 0^k 1^{2n-k}$, $H_T^*(Gr(k;2n)) \xrightarrow{\iota^*} H_T^*(SpGr(k;2n))$ *takes* S_λ *to*

$$\iota^*(S_\lambda) = \sum_\nu w\left(\right) S_\nu$$



where $w\left(\right)$

$\in H_T^*(pt) = \mathbb{Z}[y_1, \ldots, y_n]$ *is computed via* R*- and* K*-matrices in the* 5*–vertex model.*

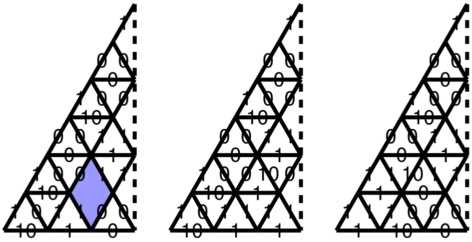
Note:

is half of a "self-dual" puzzle under Grassmann duality,

$$Gr(k;2n) \cong Gr(2n-k;2n)$$

Example and goal

Example: $\iota^*(S_{110101}) = (y_2 - y_3)S_{10,1,0} + S_{10,1,1} + S_{1,10,0}$



Goal: generalize to the 6–vertex model, understand the underlying geometry, obtain a generalized puzzle rule.

Lagrangian correspondences

A **Lagrangian correspondence** L between two symplectic manifolds A and B, $A \overset{L}{\longleftrightarrow} B$, is:

A Lagrangian cycle L in $(-A) \times B$
(equivalently L in $A \times (-B)$).

If $T \curvearrowright A, B$ and L is T–invariant, then

$$\widetilde{H}_T^*(A) \xrightarrow{(\pi_A)^*} \widetilde{H}_T^*(A \times B) \xrightarrow{\cup[L]} \widetilde{H}_T^*(A \times B) \xrightarrow{(\pi_B)_*} \widetilde{H}_T^*(B)$$

Note: In our setting, we will work with T^*G/P.

Examples

1. *Symplectic reduction*
   For $T \subseteq G \curvearrowright X$ Hamiltonian action, have a moment map $X \xrightarrow{\mu} \mathfrak{g}^*$. Take a regular value a for μ s.t. $a \in (\mathfrak{g}^*)^G$. Let $Z = \mu^{-1}(a),\ Y = \mu^{-1}(a)//G$. Then $X \hookleftarrow Z \twoheadrightarrow Y$.
   [Marsden-Weinstein '74] $\exists$! symplectic structure on Y s.t. $Z \subseteq (-X) \times Y$ is Lagrangian.

2. *Maulik–Okounkov stable envelopes*
   Suppose $S \curvearrowright X$ is a sympl. res. with a circle action.
   Let C be a fixed point component.
   The **stable envelope construction** produces a certain Lagrangian cycle $L = \overline{Attr(C)} + \ldots$ in $(-C) \times X$.

Correspondences from graphs

General setting

Let $A \xrightarrow{f} B$ be a morphism of oriented manifolds. $\Gamma(f)$=graph of f. $\Gamma(f)^{tr} \subseteq B \times A$ is a correspondence inducing $f^* : H^*(B) \to H^*(A)$.

Examples:

- Diagonal inclusion $M \xhookrightarrow{\Delta} M \times M$. Then $\Gamma(\Delta)^{tr}$ induces
$$H^*(M) \otimes H^*(M) \xrightarrow{m} H^*(M).$$
- The graph of the inclusion $Fl(j, k; n) \hookrightarrow Gr(j; n) \times Gr(k; n)$ induces multiplication
$$H^*(Gr(j; n)) \otimes H^*(Gr(k; n)) \xrightarrow{m} H^*(Fl(j, k; n)).$$
- The graph of $SpGr(k; 2n) \overset{\iota}{\hookrightarrow} Gr(k; 2n)$ induces the restriction
$$H^*(Gr(k; 2n)) \to H^*(SpGr(k; 2n)).$$

Lifting to cotangent bundles

Assume we have a torus action $T \curvearrowright A, B$. We have the following commutative diagram of correspondences. It allows us to study the bottom row in cohomology via the symplectic setting of the top row.

$$
\begin{array}{ccc}
T^*B & \xrightarrow{C(\Gamma(f))^{tr}} & T^*A \\
\Gamma(\iota_B)\uparrow & & \uparrow\Gamma(\iota_A) \\
B & \xrightarrow{\Gamma(f)^{tr}} & A
\end{array}
$$

$$
\begin{array}{ccccccc}
\widetilde{H}^*_{T\times\mathbb{C}^\times}(T^*B) & \longrightarrow & \widetilde{H}^*_{T\times\mathbb{C}^\times}(T^*A) & & \beta & \longmapsto & \alpha \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\widetilde{H}^*_{T\times\mathbb{C}^\times}(B) & \xrightarrow{f^*} & \widetilde{H}^*_{T\times\mathbb{C}^\times}(A) & & \frac{\beta}{[B\subseteq T^*B]} & \longmapsto & \frac{\alpha}{[A\subseteq T^*A]}
\end{array}
$$

Maulik–Okounkov classes

For a regular circle action $S \curvearrowright T^*G/P$ and a fixed pt. $\lambda \in W/W_P$, the stable envelope construction produces an MO cycle

$$MO_\lambda = \overline{BB}_\lambda + \sum_{\mu \le \lambda} a_{\lambda,\mu} \overline{BB}_\mu, \quad a_{\lambda,\mu} \in \mathbb{Z}_{\ge 0}$$

$BB_\lambda = Attr(\lambda) = CX^o_\lambda :=$ conormal bundle of the Bruhat cell X^o_λ.
This in turn gives a class $[MO_\lambda] \in H^*_{T\times\mathbb{C}^\times}(T^*G/P) \cong H^*_T(G/P)[\hbar]$.
Segre–Schwartz–MacPherson:

$$SSM_\lambda = \frac{[MO_\lambda]}{[\text{zero section}]} \in \widetilde{H}^0_{T\times\mathbb{C}^\times}(T^*G/P)$$

$$\Rightarrow SSM_\lambda = \hbar^{-\ell(\lambda)} S_\lambda + \text{l.o.t}(\hbar) \quad \Leftrightarrow S_\lambda = \lim_{\hbar\to\infty} \left(SSM_\lambda \cdot \hbar^{\ell(\lambda)}\right)$$

Structure constants: $\quad c^\nu_{\lambda\mu} = \lim_{\hbar\to\infty} \left((c')^\nu_{\lambda\mu} \cdot \hbar^{\ell(\lambda)+\ell(\mu)-\ell(\nu)}\right)$

The Sp_{2n} case

Theorem in progress (H–Knutson–Zinn-Justin '20)

There are Lagrangian correspondences

$$\lambda \xleftrightarrow{L_1} T^*Gr(k;2n) \xleftrightarrow{L_2} T^*OGr(k;4n) \xleftrightarrow{L_3} T^*SpGr(k;2n)$$

that compute the restriction of SSM classes, and together with the 6-vertex R- and K-matrices realize a puzzle rule.

- $L_1 = MO_\lambda$ is the stable envelope for the circle action
$$S_1 \cong Diag(t, t^2, \ldots, t^{2n}).$$
- $L_2 = Attr(T^*Gr(k;2n))$ is the stable envelope for the circle
$$S_2 \cong Diag(t^{-1}, \ldots, t^{-1}, t, \ldots, t).$$
- L_3 is obtained by symplectic reduction.

The End

Thank you!