A Generalized van Est map

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- More precisely, a van Est map is a map taking **Lie groupoid cohomology** to **Lie algebroid cohomology**.
- Examples of such geometric structures classified by cohomology are **groupoid representations** on line bundles, **groupoid morphisms** into $\mathbb{C}^*$, **equivariant gerbes**, **central extensions** of groupoids, etc. The last example is in particular relevant to **quantization of Poisson manifolds.**

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- **Crainic** (2003) completed the proof of the **isomorphism** theorem, again for coefficients in a **representation**.
- van Est maps have since been studied by **Abad**, **Cabrera**, **Drummond**, **Meinrenken**...

What This Talk is About

We will discuss the **generalization** of the van Est isomorphism theorem to Lie groupoids with coefficients in **more general sheaves**, such as $\mathcal{O}^*$. In addition, we discuss the **generalization** to **complex Lie groupoids**. arXiv:1909.12100v2

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- application
- future directions

Family of Abelian Groups

- Let A be an abelian Lie group. Then to each manifold X we have a groupoid $X \times A \rightrightarrows X$, whose source and target are the projection onto X, and such that the multiplication is induced by the multiplication on A, ie. $(x, a) \cdot (x, b) = (x, ab)$. This is an example of a **family of abelian groups**, We denote it A_X. We will call this a **trivial** family of abelian groups.

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- Definition: Let X be a manifold. A family of groups over X is a Lie groupoid $M \rightrightarrows X$ such that the source and target maps are equal. A family of groups will be called a **family of abelian groups** if the multiplication on M induces the structure of an **abelian group** on its source (or target) **fibers**.

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- My preferred examples to keep in mind are $A = \mathbb{C}^*$ or $A = S^1$.

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where $\mathbf{B}^0 G = G^0$, $\mathbf{B}^1 G = G$, and

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- For $n = 1$ the face maps are the source and target maps,
- for $(g_0, \ldots, g_n) \in \mathbf{B}^{n+1} G$ the face maps are

$$d_{n+1,0}(g_0, \ldots, g_n) = (g_1, \ldots, g_n)\,,$$
$$d_{n+1,i}(g_0, \ldots, g_n) = (g_0, \ldots, g_{i-1}g_i, \hat{g}_i, \ldots, g_n)\,,\ \ 1 \le i \le n$$
$$d_{n+1,n+1}(g_0, \ldots, g_n) = (g_0, \ldots, g_{n-1})\,.$$

Lie Groupoid Cohomology

- Given an abelian Lie Group A, we can define the **cohomology** of G with coefficients in A to be the cohomology of the **simpicial manifold $\mathbf{B}^\bullet G$**, where on $\mathbf{B}^n G$ we put the sheaf $\mathcal{O}(A_{\mathbf{B}^n G})$, ie. the **sheaf of A-valued functions**.

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Sheaves on simplicial manifolds have enough injectives.

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- In degree 1, this cohomology classifies G-**equivariant**, **principal A-bundles** on G^0 (principal A-bundles on G).
- In degree 2, this cohomology classifies G-**equivariant gerbes** on G^0.

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- Definition (Tu, 2006): Let $G(a, b)$ be the space of morphisms with source a and target b. A G-module $M \to G^0$ is a family of abelian groups, together with an action of G on M, such that $G(a, b) : M_a \to M_b$ acts by homomorphisms (here M_a, M_b are the fibers of M over a, b, respectively).

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- A nice example of a G-module which isn't a fiber bundle is what we call $\mathbb{C}^*_X(*D) \to X$, whose **sheaf of sections** is isomorphic to the **sheaf of meromorphic functions** valued in $\mathbb{C}^*$, with poles and zeros only allowed on the **divisor** $D \hookrightarrow X$.

Lie Algebroid Cohomology

- Let $\mathfrak{g} \to X$ be a **Lie algebroid** and let A be an **abelian Lie group**, with **exponential map** $\exp : \mathfrak{a} \to A$. We then define sheaves on X, called **sheaves of A-valued forms**, as follows: let

$$\mathcal{C}^0(\mathfrak{g}, A_X) = \mathcal{O}(A_X)\,,$$
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- We then have a **cochain complex** of sheaves given by

$$\mathcal{C}^0(\mathfrak{g}, A_X) \xrightarrow{d_{\text{CE}} \log} \mathcal{C}^1(\mathfrak{g}, A_X) \xrightarrow{d_{\text{CE}}} \mathcal{C}^2(\mathfrak{g}, A)_X \xrightarrow{d_{\text{CE}}} \cdots\,.$$

  The **sheaf cohomology** of the above complex of sheaves is denoted by $H^*(\mathfrak{g}, A_X)$.

Example: Deligne Cohomology

- If we let $\mathfrak{g} = TX$ and $A = \mathbb{C}^*$, the cochain complex we get is

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- In degree 1 this cohomology classifies complex **line bundles with a flat connection**.
- The initial integration (ie. the **source simply connected integration**) of TX is $\Pi_1(X)$, the **fundamental groupoid** of X. So a **van Est** theorem, stated in the right context, should tell you when a line bundle with a flat connection **integrates** to a **representation** of $\Pi_1(X)$. The answer is: **always**.

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- If the **source fibers** of $G \rightrightarrows G^0$ are n-**connected**, then a class in $H^{n+1}(\mathfrak{g}, A_X)$ is in the image of the **van Est map** if and only if the **translation** of this class to each **source fiber** is **trivial**.

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- Suppose π^{-1} has **integral periods** on $X - D$.
- Then if (G, ω) is a **source simply connected**, **symplectic groupoid** integrating the **Poisson manifold** (X, π), then ω is **prequantizable** to a $\mathbb{C}^*$**-extension** of G.

Future Directions: Double Groupoids

Joint work (in progress) with **Francis Bischoff**.

Without getting much into **double Lie groupoids**, I'll try to explain why a **souped-up** version of the **van Est map** is desired, one taking the **cohomology** of a **double Lie groupoid** to the **cohomology** of its **Lie algebroid-groupoid**.

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- This differentiates to a Lie algebroid-groupoid

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- The **van Est** theorem stated above **only** gives the **cohomology** up to degree $n+1$ if the **source fibers** are n-connected, ie. in **higher degrees** it **doesn't** tell you anything about the **cohomology** of the group(oid).

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- $K \hookrightarrow G$ is a **weak homotopy equivalence** of spaces, thus its fiber is weakly **contractible** because its **classifying space**, given by $(\mathbf{EG} \times \mathbf{G})/\mathbf{K}$, is weakly **contractible**. This implies that the **cohomology** of this **Lie algebroid-groupoid** is **isomorphic** to the **cohomology** of G, in **all degrees**.

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- This should **imply** van Est's **original result**.