On two notions of a gerbe over a stack

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16th September 2020.

Plan

1. Lie groupoids.
2. Differentiable stacks.
3. correspondence between Lie groupoids and differentiable stacks.
4. morphism of Lie groupoids and morphisms of differentiable stacks.

Lie groupoids

Definition (Lie groupoid)

A *Lie groupoid* is a groupoid $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$, where $\mathcal{G}_0, \mathcal{G}_1$ are smooth manifolds, with following conditions on structure maps:

- $s, t : \mathcal{G}_1 \to \mathcal{G}_0$ are (surjective) submersions,
- $m : \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_1$ is smooth map,
- $i : \mathcal{G}_1 \to \mathcal{G}_1$ is a smooth map,
- $u : \mathcal{G}_0 \to \mathcal{G}_1$ is a smooth map.

Example

Any manifold M can be considered as a Lie groupoid $(M \rightrightarrows M)$.

Example

Any Lie group G can be considered as a Lie groupoid $(G \rightrightarrows *)$.

Lie groupoids

Example

Any action of a Lie group G on a manifold M can be considered as a Lie groupoid $(M \times G \rightrightarrows M)$.

Similar to the case of a Lie group, we have the notion of a Lie groupoid action on a manifold, principal $\mathcal{G}$-bundle for a Lie groupoid $\mathcal{G}$, etc.

Definition (action of a Lie groupoid on a manifold)

Let $\mathcal{G}$ be a Lie groupoid and P be a manifold. A *right action of* $\mathcal{G}$ *on* P consists of a pair of smooth maps $(a_{\mathcal{G}} : P \to \mathcal{G}_0, \mu : P \times_{\mathcal{G}_0,t} \mathcal{G}_1 \to P)$ satisfying the following conditions:

1. $p \cdot 1 = p$ for all $p \in P$,
2. $a_{\mathcal{G}}(p \cdot \gamma) = s(\gamma)$ for all $(p, \gamma) \in P \times_{\mathcal{G}_0} \mathcal{G}_1$,
3. $(p \cdot \gamma) \cdot \gamma' = p \cdot (\gamma \circ \gamma')$ for all $(p, \gamma, \gamma') \in P \times_{a_{\mathcal{G}},\mathcal{G}_0,t} \mathcal{G}_1 \times_{s,\mathcal{G}_0,t} \mathcal{G}_1$.

Lie groupoids

Definition (principal $\mathcal{G}$-bundle over a manifold)

Let $\mathcal{G}$ be a Lie groupoid and M be a manifold. A *principal $\mathcal{G}$-bundle over M* consists of,

1. a smooth manifold P,
2. a right action of $\mathcal{G}$ on P given by the pair

$$(a_{\mathcal{G}} : P \to \mathcal{G}_0, \mu : P \times_{\mathcal{G}_0, t} \mathcal{G}_1 \to P),$$

3. a surjective submersion $\pi : P \to M$,

satisfying the following conditions:

1. the map $\pi : P \to M$ is $\mathcal{G}$-invariant,
2. the map $P \times_{\mathcal{G}_0, t} \mathcal{G}_1 \to P \times_{\pi, M, \pi} P$ given by $(p, \gamma) \mapsto (p, p \cdot \gamma)$ is a diffeomorphism.

“stack” of principal $\mathcal{G}$-bundles

For a manifold M, let $B\mathcal{G}(M)$ denote the category, whose objects are principal $\mathcal{G}$-bundles over the manifold M, and whose morphisms are morphisms of principal $\mathcal{G}$-bundles over the manifold M. For similar reason as in the case of morphism of principal G-bundles over a manifold M, any morphism of principal $\mathcal{G}$-bundles over a manifold M is an isomorphism. Thus, the category $B\mathcal{G}(M)$ is a groupoid for each manifold M.

Consider the “psuedo-functor” $B\mathcal{G} : \text{Man} \to \text{Gpd}$, that assigns for each manifold M the category $B\mathcal{G}(M)$, and for each morphism $M \to M'$ the functor $B\mathcal{G}(M') \to B\mathcal{G}(M')$ given by pullback of principal bundles. It turns out that this psuedo-functor is “locally determined”.

Any psuedo-functor $\mathcal{D} : \text{Man} \to \text{Gpd}$ that is locally determined is called **a stack over (the site) Man.** We call $B\mathcal{G} : \text{Man} \to \text{Gpd}$ to be *the stack of principal $\mathcal{G}$-bundles*.

stack over a site

Let $\mathcal{C}$ be a category and $\mathcal{J}$ be a Grothendieck topology on $\mathcal{C}$. Let U be an object of $\mathcal{C}$ and $\{\sigma_\alpha : U_\alpha \to U\}$ be a covering for U. Let $U_{\alpha\beta}$ denote the fiber product of morphisms $U_\alpha \to U, U_\beta \to U$. Let $U_{\alpha\beta\gamma}$ denote "the triple fibre product" of morphisms $U_\alpha \to U, U_\beta \to U$ and $U_\gamma \to U$".

Let $\mathcal{D} : \mathcal{C}^{op} \to \text{Cat}$ be a psuedo-functor. Let $\mathcal{D}_{\text{desc}}(\{U_\alpha \to U\})$, with the following description: an object in $\mathcal{D}_{\text{desc}}(\{U_\alpha \to U\})$ is given by a collection $\{\{s_\alpha\}, \{\phi_{\alpha\beta}\}\}$ where

1. s_α is an object of $\Phi(U_\alpha)$ for each $\alpha \in \Lambda$,
2. $\phi_{\alpha\beta} : pr_2^* s_\beta \to pr_1^* s_\alpha$ is an isomorphism in $\mathcal{D}(U_{\alpha\beta})$ for each $\alpha, \beta \in \Lambda$,

such that $\phi_{\alpha\beta}, \phi_{\beta\gamma}, \phi_{\beta\gamma}$ satisfying some cocycle condition **as morphisms** in $\mathcal{D}(U_{\alpha\beta\gamma})$ for each $\alpha, \beta, \gamma \in \Lambda$.

stack over a site

Let $\{\{s_\alpha\}, \{\phi_{\alpha\beta}\}\}$ and $\{\{t_\alpha\}, \{\psi_{\alpha\beta}\}\}$ be objects of $\mathcal{D}_{\text{desc}}(\{U_\alpha \to U\})$. A morphism from $\{\{s_\alpha\}, \{\phi_{\alpha\beta}\}\}$ to $\{\{t_\alpha\}, \{\psi_{\alpha\beta}\}\}$ is given by a collection $\{\theta_\alpha \colon s_\alpha \to t_\alpha\}_{\alpha\in\Lambda}$ where θ_α is a morphism in $\mathcal{D}(U_\alpha)$ for each $\alpha \in \Lambda$ such that,

$$pr_1^*(\theta_\alpha) \circ \phi_{\alpha\beta} = \psi_{\alpha\beta} \circ pr_2^*(\theta_\beta)$$

as morphisms in $\mathcal{D}(U_{\alpha\beta})$ for each $\alpha, \beta \in \Lambda$.

Given an object U of $\mathcal{C}$ and a cover $\{U_\alpha \to U\}$ of U, consider the following functor $\mathcal{D}(U) \to \mathcal{D}_{\text{desc}}(\{U_\alpha \to U\})$, defined by "restrictions".

Definition (a stack over a site)

Let $(\mathcal{C}, \mathcal{J})$ be a site. A pseudo-functor $\Phi : \mathcal{C}^{op} \to \text{Gpd}$ is said to be a stack (of groupoids) over the site $(\mathcal{C}, \mathcal{J})$ if it is "locally determined"; that is, for each object U of $\mathcal{C}$ and a covering $\{U_\alpha \to U\}$ of U, the induced functor $\mathcal{D}(U) \to \mathcal{D}_{\text{desc}}(\{U_\alpha \to U\})$ is an equivalence of categories.

differentiable stacks

We are interested in stacks $\mathcal{D}$: Man $\to$ Gpd that are "representable by Lie groupoids"; in the sense that, there exists a Lie groupoid $\mathcal{G}$ such that $\mathcal{D} \cong B\mathcal{G}$. We call stacks Man $\to$ Gpd that are representable by Lie groupoids to be **differentiable stacks**.

Definition (atlas of a stack)

Let $\mathcal{D} \to$ Man be a stack. An *atlas for the stack* $\mathcal{D}$ is given by manifold X and a morphism of stacks $X \to \mathcal{D}$ such that,

- for each manifold Y and a morphism of stacks $Y \to \mathcal{D}$, the fiber product $X \times_{\mathcal{D}} Y$ is "represented by a manifold" and the projection $X \times_{\mathcal{D}} Y \to Y$ induces a surjective submersion at the level of manifolds.

Theorem

A stack $\mathcal{D}$: Man $\to$ Gpd is a differentiable stack if and only if there exists an atlas for $\mathcal{D}$: Man $\to$ Gpd.

plan for next session

We have "seen" that for any Lie groupoid, we can associate a differentiable stack, and for each differentiable stack, we can associate a Lie groupoid.

Question

What can we say about the morphism of stacks $B\mathcal{G} \to B\mathcal{H}$ associated to a morphism of Lie groupoid $\mathcal{G} \to \mathcal{H}$?

Question

What can we say about the morphism of stacks $B\mathcal{G} \to B\mathcal{H}$ associated to a morphism of Lie groupoid $\mathcal{G} \to \mathcal{H}$ that is a Lie groupoid extension?

Question

Does every morphism of differentiable stack $\mathcal{D} \to \mathcal{C}$ come from a morphism of Lie groupoid?

plan for next session

Question

Does a morphism of differentiabe stack $\mathcal{D} \to \mathcal{C}$, that is a gerbe over the stack $\mathcal{C}$, come from a morphism of Lie groupoids?

Question

What can we say about the morphism of Lie groupoids assocaited to a morphism of differentiabe stack $\mathcal{D} \to \mathcal{C}$, that is a gerbe over the stack $\mathcal{C}$?

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