Periodic orbits on tentacular hyperboloids

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ETH, Zurich

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Junior Global Poisson Workshop

Hamiltonian dynamics

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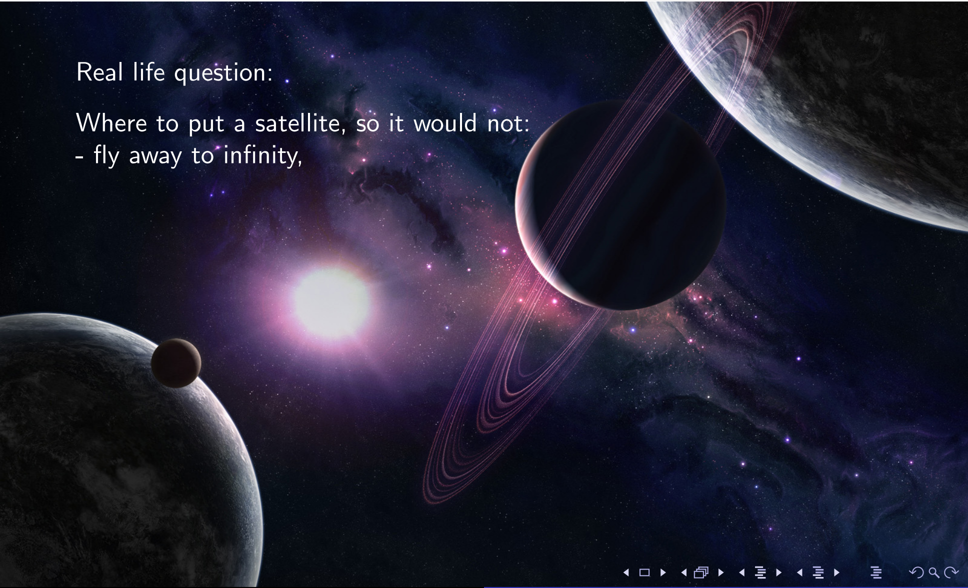
Observe: Energy is preserved by the Hamiltonian flow.
Goal: Analyze Hamiltonian flow on a fixed energy level.

Physical applications of Hamiltonian dynamics



Real life question:

Where to put a satellite, so it would not:
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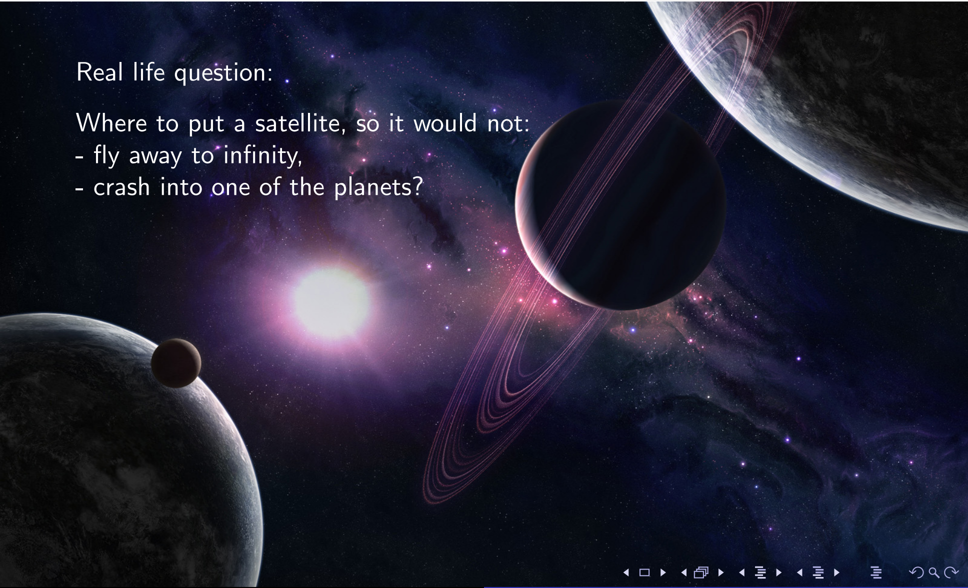

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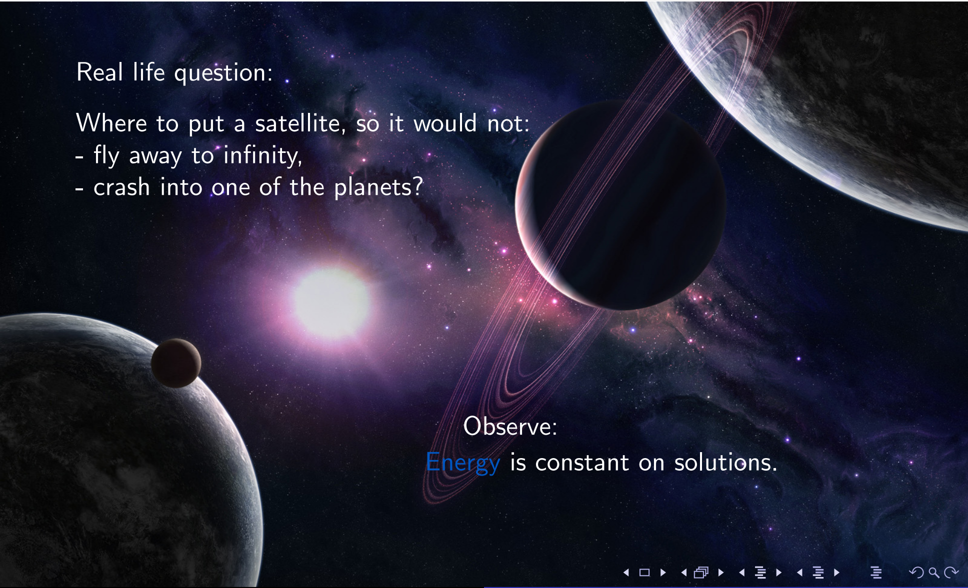

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Where to put a satellite, so it would not:
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Observe:

Energy is constant on solutions.
Energy level set might be non-compact.

Question:

Does there exist a closed orbit
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Can we relate it to the geometry
of the energy hypersurface?

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$$\mathcal{A}^H(v, \eta) := \int \lambda(\partial_t v) - \eta \int H(v)$$

$v : \mathbb{R}/\mathbb{Z} \to M, \eta \in \mathbb{R}$

Critical set

Rabinowitz action functional

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$$(v,\eta) \in \operatorname{Crit} \mathcal{A}^H \quad \Longleftrightarrow \quad \partial_t v = \eta X_H(v) \quad \text{and} \quad v(t) \in \Sigma \quad \forall\ t.$$

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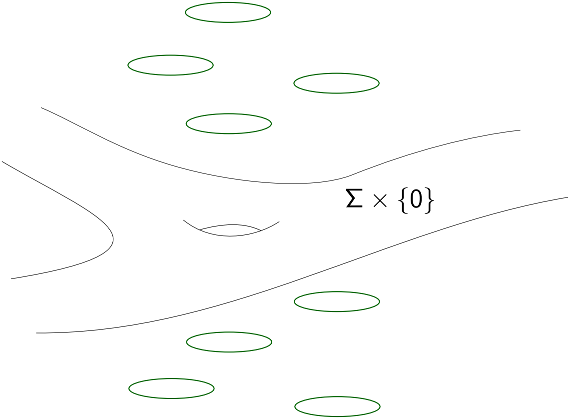

Rabinowitz Floer homology

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Build a Floer-type homology for the Rabinowitz action functional.

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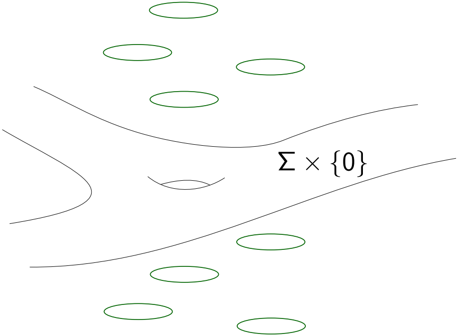
$$\begin{pmatrix} \partial_s v \\ \partial_s \eta \end{pmatrix} = \begin{pmatrix} -J(v,\eta,t)(\partial_t v - \eta X_H(v)) \\ -\int H(v) \end{pmatrix}$$

Rabinowitz Floer homology



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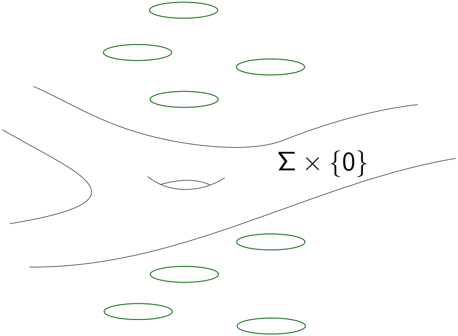

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complex - $\mathrm{Crit}(f) \otimes \mathbb{Z}_2$,

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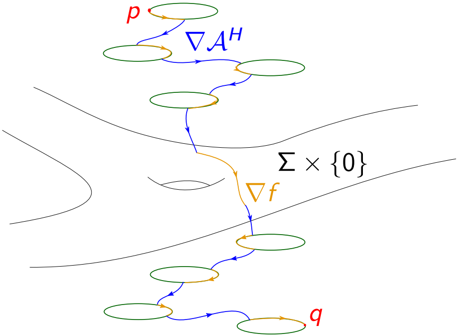

$f : \mathrm{Crit}(\mathcal{A}^H) \to \mathbb{R}$ - a Morse function

Building RFH:

complex - $\mathrm{Crit}(f) \otimes \mathbb{Z}_2$,

boundary operator - counting *cascades*.

Theorem: (Cieliebak and Frauenfelder, 2009)

Rabinowitz Floer homology is well defined for compact hypersurfaces.

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Properties of RFH:

1. If Σ_s is a smooth family of contact type, compact perturbations of Σ_0, then
$$RFH(\Sigma_s) = RFH(\Sigma_0).$$

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3. If Σ is displacable then $RFH(\Sigma) \equiv 0$.

Question:

Can we define Rabinowitz Floer homology for non-compact hypersurfaces?

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Yes!!!

Tentacular hyperboloids

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Observe: A has signature $(n+k, n-k)$.

Tentacular hyperboloids

⋆ $$H(x,y) := \frac{1}{2}\langle x, A_0 x\rangle - 1 + \frac{1}{2}\langle y, A_1 y\rangle, \qquad x \in \mathbb{R}^{2k}, \quad y \in \mathbb{R}^{2(n-k)},$$

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Definition:

We say that Σ is a tentacular hyperboloid if it is a regular level set of a Hamiltonian as in $\star$ and the Jordan decomposition of $\mathbb{J}A_1$ has for every eigenvalue λ_i an $m_i \times m_i$ block satisfying one of the following:

a) $m_i = 1$ and $|\operatorname{Re}\lambda_i| \neq 0$;
b) $m_i = 2$ and $|\operatorname{Re}\lambda_i| > \frac{1}{\sqrt{2}}$;
c) $m_i > 2$ and $|\operatorname{Re}\lambda_i| > 2$.

RFH for tentacular hyperboloids

Theorem 1: (Pasquotto, Vandervorst, W., 2017)

Rabinowitz Floer homology for tentacular hyperboloids is well defined. Moreover, it is well defined for any compact contact-type perturbation of a tentacular hyperboloid.

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Such defined RFH is:

1. Invariant under compact contact-type perturbations.
2. Equal to $H_{*+n-1}(\Sigma)$ whenever Σ has no periodic orbits.

Weinstein conjecture for tentacular hyperboloids

Theorem 2: (Fauck, Merry, W., 2020)

Let Σ be a tentacular hyperboloid defined by a Hamiltonian H as in $\star$ with $1 \leq k \leq n-1$. Then

$$RFH_*(H) = \begin{cases} \mathbb{Z}_2 & * = 1-n, -k, \\ 0 & \text{otherwise.} \end{cases}$$

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Corollary: (Weinstein conjecture)

If Σ_s is a family of compact, contact-type perturbations of a tentacular hyperboloid, then each Σ_s carries a periodic orbit.

Thank you for your attention!