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On the homotopy theory and algebraic geometry of sheaves of Lie Rinehart structures

Junior Global Poisson Seminar 2020

Joel Villatoro
KU Leuven
September 14 2020

0 Outline

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1 Outline

1 Who Am I?

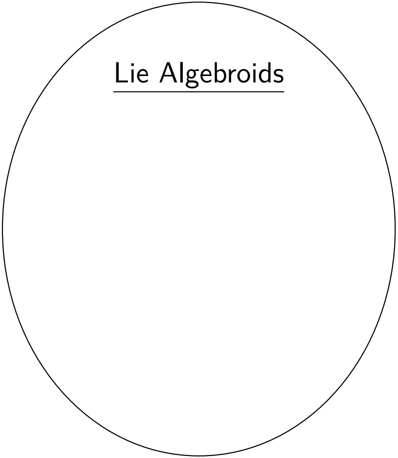
Short CV

- PhD 2018 - University of Illinois at Urbana Champaign with Rui Fernandes
- 2018 - Present - KU Leuven Postdoc with Marco Zambon

Reseach Interests

- Lie groupoids and algebroids
- Foliations
- Category theoretic perspectives on differential geometry
- Synthetic differential geometry/diffeology

1 Conceptual Map



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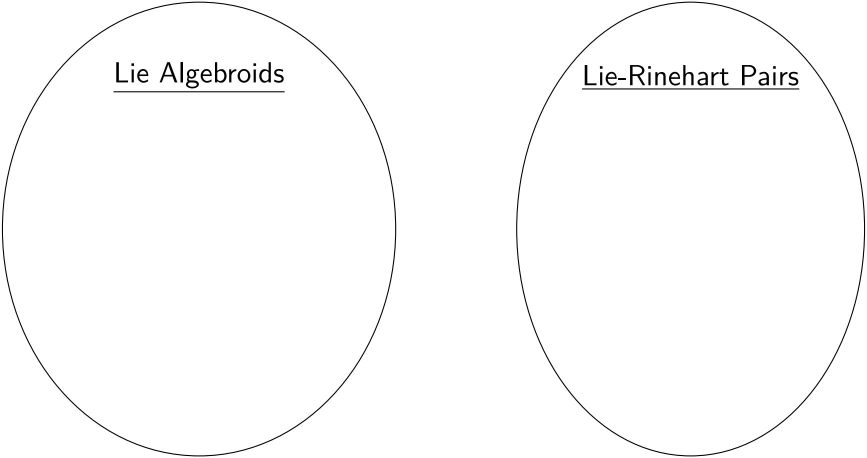

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Lie Algebroids

- Cohomology
- Singular Foliations
- Homotopy Groups
- Fundamental Groupoids

Lie-Rinehart Pairs

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Sections →

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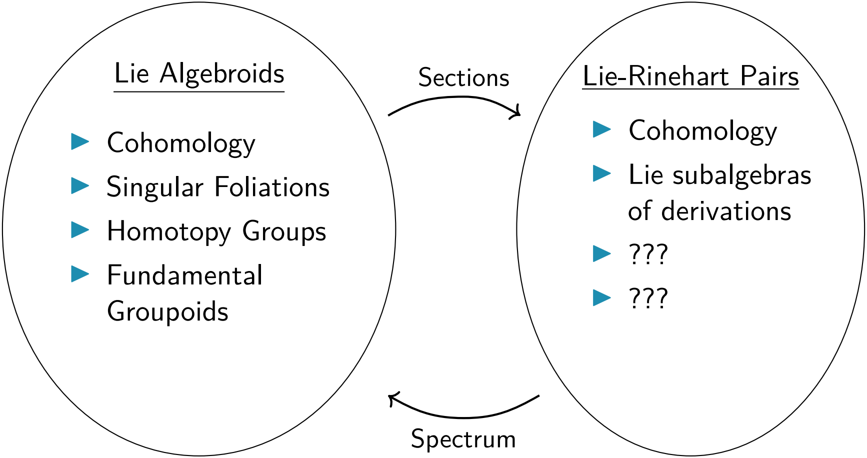

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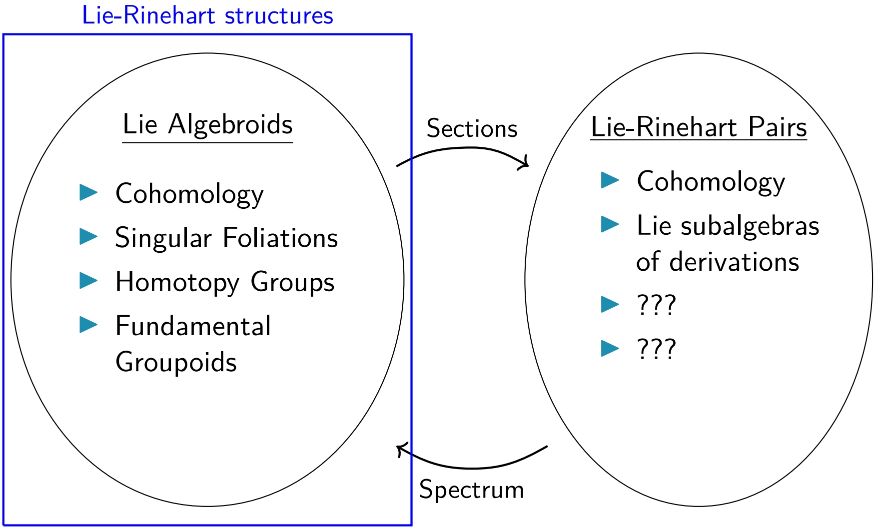

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- *Singular subalgebroids*. M. Zambon (2020).

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A **Lie algebroid** is a vector bundle $A \to M$ with a vector bundle morphism $\rho\colon A \to TM$ called the *anchor map* and a Lie bracket:

$$U \subset M \text{ open}, \quad \Gamma_A(U) \times \Gamma_A(U) \to \Gamma_A(U) \quad (\alpha, \beta) \mapsto [\alpha, \beta]$$

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such that:

- Given $U \subset M$ we have that $\rho_U : \Gamma_A(U) \to \mathfrak{X}(U)$ is a Lie algebra homomorphism,
- Given $\alpha, \beta \in \Gamma_A(U)$ and $f \in \mathcal{C}^\infty_M(U)$ we have

$$[\alpha, f\beta] = f\alpha + \mathcal{L}_{\rho(\alpha)}\beta$$

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such that:

(a) $\rho\colon \mathcal{A} \to \mathrm{Der}(R)$ is a Lie algebra homomorphism,
(b) The Leibniz rule holds:
$$\forall a, b \in \mathcal{A} \quad r \in R \qquad [a, rb] = r[a, b] + \mathcal{L}_a(r)b$$

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- Any sheaf of Lie subalgebras of Γ_A closed under the action of $\mathcal{C}^\infty_M$ (Singular subalgebroids).

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The category of LR pairs with LR morphisms is denoted LRP.

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Theorem (Higgens-Mackenzie, 1993)

Every LR morphism factors through an action LR pair.

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The category of LR pairs with LR comorphisms is LRP^{co}.

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- Given an LR pair $(\mathcal{B}, S)$ and $\phi\colon S \to R$. If $(\mathcal{A}, R)$ is the base change along ϕ, there is a canonical LR comorphism:

$$(\pi_2, \phi)\colon (A, R) \overset{co}{\to} (\mathcal{B}, S)$$

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Theorem (V?)

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$$(\pi_2, \phi)\colon (A, R) \overset{co}{\to} (\mathcal{B}, S)$$

Theorem (V?)

Every LR comorphism factors through a base change.

False for Lie Algebroids!

3 Outline

3 Lie Rinehart structures

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Lemma

The usual procedure for constructing a sheaf of modules $\mathcal{A}$ over $\operatorname{Spec}(S)$ using $\mathcal{B}$ results in a Lie Rinehart structure.

3 Morphisms and comorphisms

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4 Outline

4 LR structures on manifolds

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$$\pi_n(\mathcal{A}, x_0) := \{\text{(Pointed) Homotopy classes of } \mathfrak{X}_{\mathbb{S}^n} \to \mathcal{A}\}$$

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Corollary: Any LR structure on a manifold induces a partition into isomorphism classes.

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Suppose M is a smooth manifold and $\mathcal{A} \rightsquigarrow M$ is a Frobenius integrable Lie Rinehart structure on M. Then there exists a partition of M into immersed submanifolds whose tangent spaces agree with the span of $\rho(A) \leq \mathfrak{X}_M$.

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Theorem (V.)

Suppose $P\colon \mathcal{A} \to \mathcal{B}$ *is a fibration of Frobenius integrable LR structures covering* $p\colon E \to B$. *Let* $x_0 \in E$ *and* $K \rightsquigarrow F$ *be the fiber over* $p(x_0)$. *Then we have a long exact sequence:*

$$\ldots \to \pi_n(K, x_0) \to \pi_n(\mathcal{A}, x_0) \to \pi_n(\mathcal{B}, p(x_0)) \to \pi_{n-1}(K, x_0) \to \ldots$$

This is an extension of the analagous result of Braihic-Zhu for Lie Algebroids.

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5 Outline

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This notion of smoothness is preserved by, addition, scalar multiplication, restrictions.

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Proposition

Suppose $\alpha\colon [0,1] \to \mathcal{A}(M)$ *is smooth and* $\alpha(t) = \sum_i u^i a_i$ *is a parameterization. Then the following expressions are well-defined:*

$$\alpha'(t) = \sum_{i=1} (u^i)'(t) a_i, \quad \int_0^1 \alpha(s)\,\mathrm{d}s := \sum_i \left(\int_0^1 u^i(s)\,\mathrm{d}s \right) a_i$$

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$$\forall \beta \in \mathcal{A}(M), \qquad \frac{\mathrm{d}}{\mathrm{d}\,t}\Phi_\alpha^t(\beta) = [\alpha(t), \beta] \quad \text{"Adjoint flow equation"}$$

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Theorem (V.)

Suppose $\mathcal{A} \rightsquigarrow M$ is Frobenius integrable and locally finitely generated. Then $\mathcal{A}$ is Frobenius integrable.

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- If $\{\mathcal{E}_i\}_{i \in I}$ is an arbitrary family of Frobenius integrable LR substructures of $\mathcal{A}$ such that $\mathcal{A} = \mathrm{Span}(\{E_i\})$. Then $\mathcal{A}$ is Frobenius integrable.

6 Outline

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- Frobenius integrable LR structures are very Lie algebroid-like!

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- If $\mathcal{A}$ is locally finitely generated then the restriction of $\mathcal{A}$ to a leaf L is an actual Lie algebroid and $\Pi_1(\mathcal{A})_L$ is the set-theoretic "Weinstein groupoid" of a transitive algebroid .

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Given a time-dependent element $\beta(t)$ of $\mathcal{B}(B)$ a connection yields a time-dependent element $\sigma(\beta(t))$ of $\mathcal{A}(E)$. The connection is complete if the flow of $\sigma(\beta)$ exists as long s the flow of β exists.

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Thanks!