Lie groupoids and Logarithmic connections

Francis Bischoff

Exeter College, University of Oxford

Junior Global Poisson Workshop 2020

Support for research

Natural Sciences and Engineering Research Council of Canada
Conseil de recherches en sciences naturelles et en génie du Canada
Canada

Table of Contents

Plan of talk

- Study flat connections on principal bundles with logarithmic singularities, using tools from the theory of Lie groupoids.
- Setting: G is a complex reductive group, e.g. $G = GL(n, \mathbb{C})$.
- We study connections on principal G-bundles $p : P \to M$.

Table of Contents

Riemann-Hilbert correspondence for smooth connections

$$\left\{ \begin{array}{l} \text{Flat connections } (P, \nabla) \\ \text{on the manifold } M \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Representations of the} \\ \text{fundamental group } \pi_1(M, x) \end{array} \right\}$$

Riemann-Hilbert correspondence for smooth connections

This equivalence of categories arises by combining two more basic equivalences

$$\mathrm{Rep}(TM, G) \xleftrightarrow{\text{Lie 2}} \mathrm{Rep}(\Pi(M), G) \xleftrightarrow{\text{Morita}} \mathrm{Rep}(\pi_1(M, x), G)$$

Flat connections = Representations of TM

- Given principal G bundle $p : P \to M$, define the Atiyah algebroid $At(P) = TP/G$.
- Flat connections are representations of the tangent bundle TM:

$$\nabla : TM \to At(P),$$

  such that

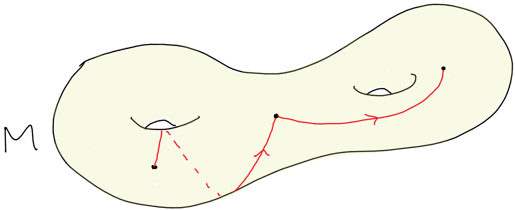
$$\nabla([X, Y]) = [\nabla(X), \nabla(Y)],$$

  and such that $dp \circ \nabla = id$, where $dp : At(P) \to TM$.
- TM and $At(P)$ are examples of Lie algebroids, and a flat connection is a morphism of Lie algebroids.
- Analogy: Representation of a Lie algebra $\phi : \mathfrak{g} \to \mathfrak{gl}(n)$.

Integration

The Lie algebroids TM and $At(P)$ can be integrated to Lie groupoids:

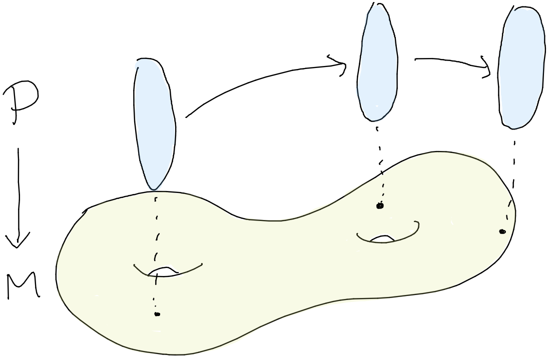
- $TM \rightsquigarrow \Pi(M)$, the fundamental groupoid of M,



Integration

The Lie algebroids TM and $At(P)$ can be integrated to Lie groupoids:

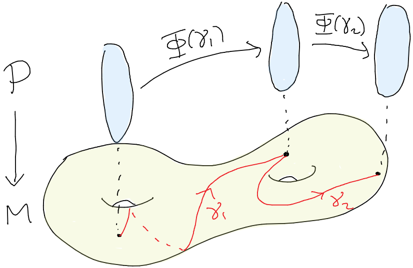
- $TM \rightsquigarrow \Pi(M)$, the fundamental groupoid of M,
- $At(P) \rightsquigarrow \mathcal{G}(P) = (P \times P)/G$, the gauge groupoid of P.



Groupoid representation

A representation of $\Pi(M)$ on a principal bundle P is a Lie groupoid homomorphism

$$\Phi : \Pi(M) \to \mathcal{G}(P).$$



Analogy: Representation of a Lie group $\Phi : G \to GL(n)$.

Lie's Second theorem

Theorem (Mackenzie-Xu, Moerdijk-Mrčun)

Let $\mathcal{G}$ be a source simply connected Lie groupoid, with Lie algebroid A. There is an equivalence of categories

$$\mathrm{Rep}(A, G) \cong \mathrm{Rep}(\mathcal{G}, G).$$

Therefore, we get the first part of the Riemann-Hilbert correspondence

$$\mathrm{Rep}(TM, G) \xleftrightarrow{\text{Lie 2}} \mathrm{Rep}(\Pi(M), G)$$

Morita equivalence

The groupoids $\Pi(M)$ and $\pi_1(M, x)$ are Morita equivalent.

- The quotient spaces $M/\Pi(M)$ and $\{x\}/\pi_1(M, x)$ are 'the same'.
- Representations of $\Pi(M)$ are a model for principal bundles over $M/\Pi(M)$, and similarly for $\pi_1(M, x)$.
- Therefore, $\Pi(M)$ and $\pi_1(M, x)$ have equivalent categories of representations.

Morita equivalence

Definition

A Morita equivalence between $\mathcal{G}$ and $\mathcal{H}$ induces an equivalence of categories

$$\mathrm{Rep}(\mathcal{G}, G) \cong \mathrm{Rep}(\mathcal{H}, G)$$

Therefore, we get the second part of the Riemann-Hilbert correspondence

$$\mathrm{Rep}(\Pi(M), G) \xleftrightarrow{\text{Morita}} \mathrm{Rep}(\pi_1(M, x), G)$$

Table of Contents

Logarithmic connections with Fuchsian singularities

Logarithmic flat connections are the differential geometric generalization of linear ODEs with Fuchsian singularities.

$$z \frac{ds}{dz} = A(z)s,$$

where $A : \mathbb{A} \to \mathfrak{g}$, and $s : \mathbb{A} \to G$ is a fundamental solution.

Normal forms and classification results due to Levelt, Turrittin, Babbitt and Varadarajan, Kleptsyn and Rabinovich, Boalch, Deligne, Simpson, Ogus, etc.
In this talk: completely functorial Riemann-Hilbert correspondence in terms of generalized monodromy data.

Lie theoretic definition

Let X be a complex manifold, with a smooth hypersurface D.

- $T_X(-\log D)$: Lie algebroid of vector fields on X which are tangent to D.
- A flat connection with logarithmic singularities along D is a Lie algebroid homomorphism

$$\nabla : T_X(-\log D) \to At(P).$$

Integration

The Lie algebroid $T_X(-\log D)$ has source simply connected integration $\Pi(X, D)$: the twisted fundamental groupoid. Therefore, applying Lie 2, we get the first part of a Riemann-Hilbert correspondence:

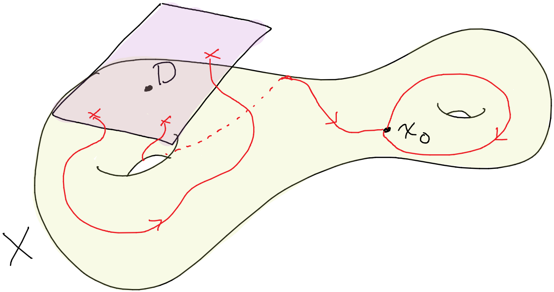
$$\mathrm{Rep}(T_X(-\log D), G) \overset{\text{Lie 2}}{\longleftrightarrow} \mathrm{Rep}(\Pi(X, D), G)$$

Morita Equivalence

$$\mathrm{Rep}(T_X(-\log D), G) \overset{\text{Lie 2}}{\longleftrightarrow} \mathrm{Rep}(\Pi(X, D), G) \overset{\text{Morita}}{\longleftrightarrow} \ ??$$

Groupoid of paths with tangential basepoints

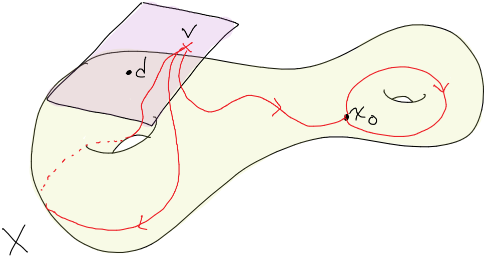
The twisted fundamental groupoid $\Pi(X, D)$ is Morita equivalent to a groupoid $\mathcal{N} \rightrightarrows N_X(D)|_d \sqcup \{x_0\}$ of paths with tangential basepoints (Deligne).



$$\mathrm{Rep}(\Pi(X, D), G) \xleftrightarrow{\text{Morita}} \mathrm{Rep}(\mathcal{N}, G)$$

Subgroupoids of $\mathcal{N}$

- Choose non-zero $v \in N_X(D)$, and consider
$$\Pi(X \setminus D)|_{\bar{v}} := \mathcal{N}|_{\{v, x_0\}},$$
which is Morita equivalent to $\pi_1(X \setminus D)$.
- $A(N_X(D))$: Paths that lie fully in the normal bundle.
- $A(N_X(D))$ is essentially the same as $\mathbb{C} \ltimes N_X(D)|_d$.



Van Kampen Theorem

Theorem

Pushout of holomorphic Lie groupoids

$$\begin{array}{ccc} \pi(N_X(D)^\times, v) & \longrightarrow & \Pi(X \setminus D)|_{\bar{v}} \\ \downarrow & & \downarrow \\ A(N_X(D)) & \longrightarrow & \mathcal{N} \end{array}$$

Upshot: Only need to study representations of $\pi_1(X \setminus D, x_0)$ and $\mathbb{C} \ltimes \mathbb{A}$.

Table of Contents

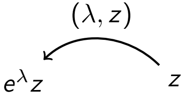
Representations of $\mathbb{C} \ltimes \mathbb{A}$

The representations of $\mathbb{C} \ltimes \mathbb{A}$ are equivalent to linear ODEs with Fuchsian singularities.

$$z\frac{ds}{dz} = A(z)s.$$

Action groupoid $\mathbb{C} \ltimes \mathbb{A} \rightrightarrows \mathbb{A}$

- $\mathbb{C} \ltimes \mathbb{A}$: action groupoid associated to the exponentiated action of $\mathbb{C}$ on $\mathbb{A}$.



- Two orbits: $\{0\}$ and $\mathbb{A} \setminus \{0\}$.
- Isotropy groups: $\mathbb{C} \to 0$ and $\mathbb{Z} \to 1$.

Classifying invariants of a representation (P, Φ)

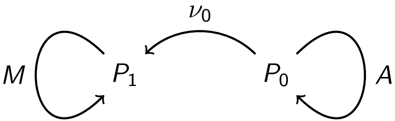
We can extract the following classifying invariants from a representation

$$\Phi : \mathbb{C} \ltimes \mathbb{A} \to \mathcal{G}(P).$$

1. Monodromy $M = \Phi(2\pi i, 1) \in Aut_G(P_1)$,
2. Residue $A \in \mathfrak{aut}_G(P_0)$, where $\Phi(\lambda, 0) = \exp(\lambda A)$,
3. Space of regularized parallel transport maps $\nu_0 = \{T : P_0 \to P_1\}$.

Classifying invariants of a representation (P, Φ)

Think of (M, A, ν_0) as generalized monodromy data:



Linear approximation

To define the regularized parallel transport, we use the concept of linear approximations. There is a natural groupoid homomorphism

$$\pi : \mathbb{C} \ltimes \mathbb{A} \to \mathbb{C} \ltimes \mathbb{A}, \qquad (\lambda, z) \mapsto (\lambda, 0).$$

The pullback defines a linear approximation functor

$$L = \pi^* : \mathrm{Rep}(\mathbb{C} \ltimes \mathbb{A}, G) \to \mathrm{Rep}(\mathbb{C} \ltimes \mathbb{A}, G).$$

This functor takes an arbitrary representation (P, Φ) and outputs the trivial representation determined by its residue

$$(\mathbb{A} \times P_0, L(\Phi) = \exp(\lambda A)).$$

Linearization

Definition

A linearization of a representation is an isomorphism

$$T : (P_0 \times \mathbb{A}, L(\Phi)) \to (P, \Phi).$$

The linearization is strict if $T(0) = id$.

- Can be thought of as a regularized parallel transport

$$T(1) : P_0 \to P_1.$$

- Linearizations encode the asymptotic nature of fundamental solutions at the singularity, and hence are closely related to the Levelt filtration.

Resonance

- Because of resonance, linearizations are not guaranteed to exist.
- However, a representation is linearizable if it has semisimple monodromy.
- Therefore, we use a Jordan Chevalley decomposition to decompose representations into semisimple and unipotent components.

Jordan Chevalley decomposition

Theorem

Let (P, Φ) be a representation, and let U denote the unipotent part of its monodromy. Then the following defines a unipotent groupoid 1-cocycle

$$\sigma_{\Phi}(\lambda, z) = \exp(\frac{-\lambda}{2\pi i} log(U(e^{\lambda} z))).$$

The deformed representation

$$\Phi_s := \sigma_{\Phi} \circ \Phi,$$

has semisimple monodromy.

This defines a functorial Jordan Chevalley decomposition for representations.

Regularized parallel transport

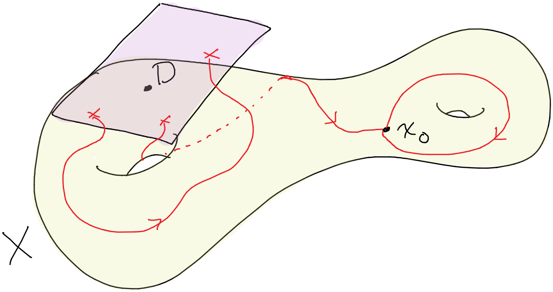
The space of regularized parallel transports ν_0 of a representation (P, Φ) is the space of strict linearizations of the semisimple component Φ_s.

$$T : (\mathbb{A} \times P_0, L(\Phi_s)) \to (P, \Phi_s).$$

Table of Contents

Riemann-Hilbert correspondence

- $\mathrm{Rep}(T_X(-\log D), G) \xleftrightarrow{\text{Lie 2}} \mathrm{Rep}(\Pi(X, D), G) \xleftrightarrow{\text{Morita}} \mathrm{Rep}(\mathcal{N}, G)$
- $\mathcal{N}$ is the groupoid of paths with tangential basepoints.

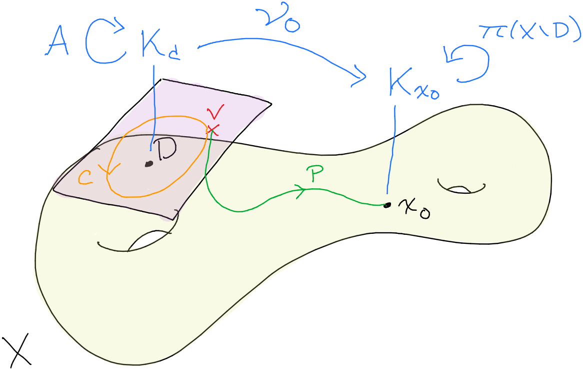


Riemann-Hilbert correspondence

- $\mathrm{Rep}(T_X(-\log D), G) \xleftrightarrow{\text{Lie 2}} \mathrm{Rep}(\Pi(X, D), G) \xleftrightarrow{\text{Morita}} \mathrm{Rep}(\mathcal{N}, G)$
- $\mathcal{N}$ is the groupoid of paths with tangential basepoints.
- $\mathcal{N}$ decomposes into building blocks $\pi_1(X \setminus D)$ and $\mathbb{C} \ltimes \mathbb{A}$.
- Combining all the steps gives a functorial Riemann-Hilbert correspondence in terms of generalized monodromy data.

Category of generalized parallel transport data

Category $F((X, D), G)$ with objects $(\Phi, K_{x_0}, \nu_0, K_d, A)$



Riemann Hilbert correspondence

Theorem

There is an explicit equivalence of categories

$$\mathrm{Rep}(T_X(-\log D), G) \cong F((X, D), G).$$

$$\left\{ \begin{array}{l} \text{Flat connections } (P, \nabla) \text{ on } X \\ \text{with log singularities along } D \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Generalized monodromy} \\ \text{data } (\Phi, K_{x_0}, \nu_0, K_d, A) \end{array} \right\}$$

Thank You