UNIVERSITY OF COPENHAGEN

Junior Global Poisson Workshop 2020

Quantization of Coadjoint Orbits

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14.09.2020

The quantization problem

Classical Physics
$(C^\infty(M), \{\,\cdot\,,\,\cdot\,\})$

Poisson manifold M

Observables: real-valued smooth functions $C^\infty(M, \mathbb{R})$

Time evolution:

$\frac{\mathrm{d}}{\mathrm{d}t} f = \{H, f\}$

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Formal expansion of strict quantization ("Taylor series")
Can be used to formulate index theorems algebraically

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Classical Physics —algebraic→ **Formal deformation quantization** —analytic→ Quantum Mechanics

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Formal deformation quantization

Definition (BFFLS, Formal star products)

Let (M, π) be a Poisson manifold. A *formal star product* on M is defined by a $\mathbb{C}[\![\nu]\!]$-bilinear associative product

$$\star \colon C^\infty(M)[\![\nu]\!] \times C^\infty(M)[\![\nu]\!] \to C^\infty(M)[\![\nu]\!]$$

such that, when written in the form $f \star g = \sum_{r=0}^{\infty} \nu^r C_r(f, g)$ with $C_r \colon C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ and $f, g \in C^\infty(M)$, then

1. C_r are bidifferential operators,
2. $C_0(f, g) = fg$ and $C_1(f, g) - C_1(g, f) = \mathrm{i}\{f, g\}$,
3. $1 \star f = f \star 1 = f$.

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Theorem (Kontsevich)

Formal star products exist on any Poisson manifold M.

Obstructions to equivariant strict quantization

Theorem (Rieffel)

Let $\star$ be a product on $C^\infty(\mathbb{S}^2)$, for which there is an involution and a C^*-norm such that the action of $\mathrm{SO}(3)$ is by isometric *-automorphisms. Then $\star$ is commutative.

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So an equivariant quantization of $\mathbb{S}^2 \simeq \mathbb{CP}^1$ with C^*-algebras containing $C^\infty(\mathbb{S}^2)$ is not possible.

Towards strict quantization

Possible approach (Beiser–Waldmann):

- Start with a formal deformation quantization $\star$ of a Poisson manifold M.
- Find a subalgebra $\mathcal{P} \subseteq C^\infty(M)$ ("polynomials") on which the power series defining $\star$ are finite.
- Replace ν by $\hbar \in \mathbb{C}$ to obtain products $\star_\hbar \colon \mathcal{P} \times \mathcal{P} \to \mathcal{P}$.
- Find a topology on $\mathcal{P}$ with respect to which $\star_\hbar$ is continuous and extend $\star_\hbar$ to the completion.

Function algebras on $\mathbb{CP}^n$

Consider the following algebras:

$$\begin{array}{ccc} \mathrm{Pol}^H(\widehat{\mathbb{CP}^n}) & \subseteq & \mathrm{Hol}(\widehat{\mathbb{CP}^n}) \\ \Delta^* \downarrow \simeq & & \simeq \downarrow \Delta^* \\ \mathrm{Pol}(\mathbb{CP}^n) & \subseteq & \mathcal{A}(\mathbb{CP}^n) \end{array}$$

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Recall that $\mathbb{CP}^n = \{[z] \mid z \in \mathbb{C}^{1+n} \backslash \{0\}\}$ and define

$$\mathrm{Pol}(\mathbb{CP}^n) = \{f \in C^\infty(\mathbb{CP}^n) \mid \exists p \in \mathrm{Pol}(\mathbb{C}^{1+n})^{\mathrm{U}(1)} \text{ s.t. } f([z]) = p(z) \text{ for all } z \in \mathbb{S}^{1+2n} \subseteq \mathbb{C}^{1+n}\}.$$

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Let $\widehat{\mathbb{CP}}^n = \{([x],[y]) \mid x, y \in \mathbb{C}^{1+n} \text{ and } \sum_{i=0}^n x^i y^i \neq 0\} \subseteq \mathbb{CP}^n \times \mathbb{CP}^n$.

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Lemma

Every holomorphic function f on $\widehat{\mathbb{CP}}^n$ is uniquely determined by its restriction $\Delta^* f$ to $\mathbb{CP}^n$.

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The image $\mathcal{A}(\mathbb{CP}^n) = \{\Delta^* f \mid f \in \mathrm{Hol}(\widehat{\mathbb{CP}}^n)\}$ is isomorphic to $\mathrm{Hol}(\widehat{\mathbb{CP}}^n)$ as an algebra.

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Consider $\mathcal{A}(\mathbb{CP}^n)$ with the topology corresponding to the topology of locally uniform convergence on $\mathrm{Hol}(\widehat{\mathbb{CP}}^n)$.

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Lemma

$\mathrm{Pol}(\mathbb{CP}^n)$ is a dense subset of $\mathcal{A}(\mathbb{CP}^n)$.

$\mathrm{SU}(1+n)$-equivariant quantization of $\mathbb{CP}^n$

Theorem (S–Schötz, S, Kraus–Roth–Schötz–Waldmann)

For every $\hbar \in \mathbb{C}\backslash\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ *there is a* $\mathrm{SU}(1+n)$*-equivariant product*

$$\star_\hbar \colon \mathcal{A}(\mathbb{CP}^n) \times \mathcal{A}(\mathbb{CP}^n) \to \mathcal{A}(\mathbb{CP}^n)$$

making $\mathcal{A}(\mathbb{CP}^n)$ *a Fréchet algebra.*

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- *For all* $p, q \in \mathrm{Pol}(\mathbb{CP}^n)$*, we have*

$$\lim_{\hbar \to 0} p \star_\hbar q = pq \quad \text{and} \quad \lim_{\hbar \to 0} \frac{1}{\hbar}(p \star_\hbar q - q \star_\hbar p) = \mathrm{i}\{p, q\}\,.$$

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- *For all* $f, g \in \mathcal{A}(\mathbb{CP}^n)$ *and* $z \in \mathbb{CP}^n$*, the function* $\hbar \mapsto f \star_\hbar g(z)$ *is holomorphic.*

Equivariant quantization of coadjoint orbits

Theorem (S)

Let $\mathcal{O}_\lambda$ be a semisimple coadjoint orbit of a semisimple connected Lie group G. There is a countable set of poles P, accumulating only at 0, and for every $\hbar \in \mathbb{C} \setminus P$ there is a G-equivariant product

$$\star_\hbar \colon \mathcal{A}(\mathcal{O}_\lambda) \times \mathcal{A}(\mathcal{O}_\lambda) \to \mathcal{A}(\mathcal{O}_\lambda)$$

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- *For all $f, g \in \mathcal{A}(\mathcal{O}_\lambda)$ and $\xi \in \mathcal{O}_\lambda$, the function $\hbar \mapsto f \star_\hbar g(\xi)$ on $\hbar$ is holomorphic.*

Summary

Work with the **extended space** $\widehat{\mathbb{CP}}^n$ instead of $\mathbb{CP}^n$ (with the **complexification** $\hat{\mathcal{O}}_\lambda$ instead of $\mathcal{O}_\lambda$):

- Alekseev–Lachowska constructed a product on $\mathrm{Hol}(\widehat{\mathbb{CP}}^n)[[\nu]]$ (on $\mathrm{Hol}(\hat{\mathcal{O}}_\lambda)[[\nu]]$).

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Wick rotation:

Quantizations of different real orbits with the same complexification are isomorphic as Fréchet algebras (not necessarily as Fréchet *-algebras!)

Relation to Berezin quantization

Theorem (Karabegov, Esposito–S–Waldmann)

Let $\hbar = \frac{1}{n}$ for some $n \in \mathbb{N}$. Then the product $\star_\hbar$ on $\mathbb{CP}^n$ stays well-defined on polynomials of degree $\leqslant n$, and these algebras coincide with Berezin quantization.

The algebras at the poles are therefore intimately related to finite dimensional representations of $\mathrm{SU}(1+n)$, and the Fréchet algebras interpolate between these finite dimensional algebras.