Deformation theory

Deformation problem ⇝ DGLA, MC elements $\left(da + \frac{1}{2}[a,a] = 0\right)$

L_∞-algebra? $(d, [\cdot,\cdot], l_3(\cdot,\cdot,\cdot), l_4(\cdot,\cdot,\cdot,\cdot), \ldots)$

<u>Small defos of coisotropic submfds:</u> (symp Oh, Park, Poisson: Schätz, Zambon)

$E := \ker \omega_Y$

$NY \overset{\omega}{\cong} E^*$

Equip E^* with str of L_∞-algebroid

small defos of Y $\overset{1:1}{\longleftrightarrow}$ Sections of E^* which are Maurer-Cartan elements

Forgetful map: {deformation of coisotropic A-brane (Y, F)} ⟶ {deformations of coisotr submfd}

Deformations of some coisotropic branes

[I] Lagrangian submfds (Exercise!)

$(Y, F) = (Y, 0)$

L_∞-alg str on $\mathcal{E}^*$ trivial, only d

defos $\overset{1:1}{\longleftrightarrow}$ $\Omega^1_{cl}(Y)$

defos / Ham isotopy $\cong H^1(Y)$

brane defos = coisotropic defos

forgetful map = identity

(II) Spacefilling branes $M = Y \rightsquigarrow$ Hol symp. $(M, \Omega = F + i\omega)$

Y stays the same, only F can be deformed

forgetful map $\{\text{defos of } F\} \longrightarrow \{pt\}$

M 4-mfld

$\ker \Omega = TM^{0,1}$

$\Omega \in \Omega^{2,0}(M)$, $[\Omega]$ det up to scaling

Local Torelli thm: $\{\text{small defos of } I\}$
$\updownarrow$ 1:1
$\{\text{small defos of } [\Omega]\}$

Example and Counterexample

On $M = N \times S^1 \times \mathbb{R}$, (N, Ω) hol symp., $\omega = \mathrm{Im}(\Omega) + dq \wedge dp$

$q \in S^1$, $p \in \mathbb{R}$

$Y = N \times S^1 \times \{0\}$ $E = \langle \frac{\partial}{\partial q} \rangle$

Coisotropic defor: Any $\{(p, q, f(p, q)) \mid p \in N, s \in S^1, f: Y \to \mathbb{R} \text{ smooth}\}$

(codim.-1 submfds of (M, ω) are coisotropic)

But clearly not all of them are branes!

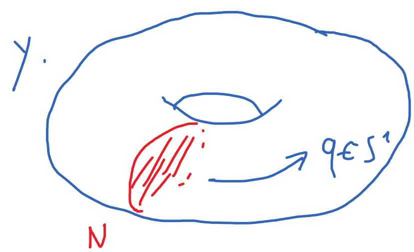
Fix $f \in C^\infty(Y)$, assume that $Y_f = \text{graph}(f)$ does admit brane str with 2-form $\tilde{F}$.

$Y_f \cong Y$

Presymp form: $\omega^f := \iota_f^* \omega = \omega_Y - d(f dq)$

<u>Denote:</u> $\omega_N := \iota^*_{N \times \{0\} \times \{0\}} \omega$ ← pullback of ω^f to $N \times \{0\}$ always ω_N

$\tilde{F}_N := \iota^*_{N \times \{0\} \times \{0\}} \tilde{F}$



Example and Counterexample (II)

Char dist. $E^f = \langle \frac{\partial}{\partial q} - X_f^{\omega_N} \rangle$

└ Harm. vf of $f|_{N \times \{q\}}$

<u>Facts about forms on foliated mfds</u>

Y mfd, $F \in \Omega^2(Y)$ const rank

Assume: $E = \ker(F)$ has involutive complement G in TY (strong assumption!)

Then F is closed iff

- E is involutive
- $\iota_G^* F$ is closed
- $\forall X \in \Gamma(E)$ s.t. $[X, \Gamma(G)] \subseteq \Gamma(G)$, the flow of X preserves $\iota_G^* F$.

Example and Counterexample (III)

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Thus if $\tilde{F}$ 2-form on Y with kernel $\mathcal{E}^f$, $d\tilde{F}=0$ iff

- $\forall q \in S^1$, $\iota^*_{N\times\{q\}}\tilde{F}$ is closed
- The flow of $\frac{\partial}{\partial q} - X_f^{\omega_N}$ preserves family of 2-forms $\iota^*_{N\times\{q\}}\tilde{F}$

So if $d\tilde{F}=0$, $\tilde{F}$ is det. by pullback to slice $N\times\{0\}$.

View $X_{f(q)}^{\omega_N} \in \Gamma(TY)$ as time-dep vf $X_{f_q}^{\omega_N}$ on $N\times\{0\}$ (q time-coord)

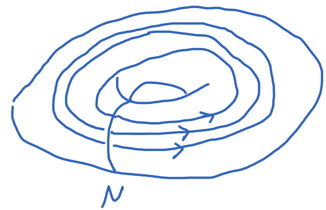
$\Rightarrow$ Time-1 flow of $X_{f_q}^{\omega_N}$ preserves $\tilde{F}_N$ (condition on $N\times\{0\}$ only!)

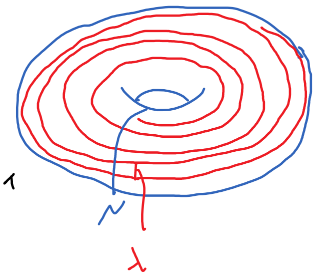
Bijection: $\left\{\begin{array}{l}\text{presymp forms } \tilde{F} \text{ on } Y \\ \text{with kernel } \mathcal{E}_f\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{l}\text{sympl forms } \tilde{F}_N \text{ on } N \text{ preserved by} \\ \text{time-1 flow of } \{X_{f_q}^{\omega_N}\}_{q\in[0,1]}\end{array}\right\}$

Example and Counterexample (IV)

Example: Take $\tilde{F}_N = F_N$

⇝ 2nd-degree PDE for f_q





Plenty of solutions n examples, e.g. take

$(N, \Omega) = \mathbb{C}^* \times \mathbb{C} \cong S^1 \times \mathbb{R}^3$

$\omega_N = dx^1 \wedge dy_1 + dx^2 \wedge dy_2$, $F_N = -dx_1 \wedge dx_2 + dy_1 \wedge dy_2$

(angle word)

Set $f(x^1, y_1, x^2, y_2, q) = \lambda y_1$. Then $\frac{\partial}{\partial q} - X_f^{\omega_N} = \frac{\partial}{\partial q} - \lambda \frac{\partial}{\partial x^1}$

$\lambda \in \mathbb{R}$ Flow preserves ω_N and F_N

If $\lambda \in \mathbb{R} \setminus \{0\}$: non-compact orbits!

Example and Counterexample (V)

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Non-example

Assume N is a K3 surface, Graph(f) as above Shane.

Defn: A K3 surface is a (compact) complex surface N with $H^1(N)=0$ and a nowhere vanishing hol. 2-form $\Omega \in \Omega^{2,0}(N)$. (Triv canon bundle)$_h$

Facts:
- All K3 surfaces are diffeom. as smooth mflds
- Local Torelli: complex str $\longleftrightarrow$ $[\Omega]$
- K3 surfaces have finitely gen group of automs

Thm: ϕ an autom. of K3 N st. $\phi^* = \mathrm{Id}$ on $H^2(N,\mathbb{Q})$, then $\phi = \mathrm{Id}$

Example and Counterexample (VI)

$\phi = \phi^1_{X_t}$ is time-1 flow of time-dep. vf on N, so isotopic to id.

$\Rightarrow \phi^* |_{H^1(N,\mathbb{R})} = \mathrm{id}$

$\Rightarrow \phi = \mathrm{id}$

Clearly there are many X_t s.t. $\phi^1_{X_t} = \mathrm{id}_N$!

Corollary: $\forall \varepsilon > 0 \ \exists$ coisotropic submfd ε-close to Y which is not a graph

Spacefilling branes on 4-manifolds

(M^4, ω), $(Y, \mathcal{F}) = (M, \mathcal{F})$ ← symplectic

$\Omega = \mathcal{F} + i\omega$ hol symp form

4-mflds
- $\mathbb{C}^2/\Lambda$ complex tori
- K3 surfaces

$\mathcal{F} \rightsquigarrow \mathcal{F}'$, ω same

$\Omega' = \mathcal{F}' + i\omega$

$[\Omega'] \wedge [\Omega'] = 0$ → $[\mathcal{F}] \wedge [\omega] = 0$

$[\mathcal{F}]^2 = [\omega]^2$

→ $[\Omega'] \wedge [\bar{\Omega}'] \neq 0$

$[\alpha] \in H^2(Y)$

$d\Omega' = 0$

⟹ $[\alpha] \wedge [\omega] = 0$

$[\alpha] \wedge [\alpha] + [\mathcal{F}] \wedge [\alpha] = 0$