# Coisotropic A-branes in Symplectic Manifolds

14.09.20, Junior Global Poisson Workshop

Coisotropic A-branes in Symplectic Manifolds

C. Kirchhoff-Lukat

(KU Leuven)

joint w. M. Gualtieri, M. Zambon

## Introduction

$M$ smooth (& frequently compact). $H \in \Omega^3_{cl}(M)$ (Frequently $[H]=0$)

- Generalized complex geometry (Hitchin, Gualtieri, Cavalcanti)

<u>Defn.</u> $\mathcal{J} \in O(TM \oplus T^*M, \langle \cdot,\cdot \rangle)$ is GC if

- $\mathcal{J}^2 = -\mathrm{Id}$
- $(+i)$-eigenbundle $L \subseteq T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ is closed under

$$[\![ X+\xi, Y+\eta ]\!]_H = [X,Y] + \mathcal{L}_X \eta - i_Y d\xi - i_Y i_X H$$

<u>Ex</u> $(M,\omega)$ symp. $(H=0)$

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \text{ is GC.}$$

$TM \oplus T^*M$

$$L = \{ X - i\omega(X) \mid X \in T_{\mathbb{C}}M \}$$
$$= \{ X + i\omega^{-1}(\xi) + \xi - i\omega(X) \mid X+\xi \in TM \oplus T^*M \}$$
$$= \{ \mathbb{X} - i\mathcal{J}_\omega \mathbb{X} \mid \mathbb{X} \in TM \oplus T^*M \}$$

(Automorphisms)

Generalized diffeomorphisms: $\mathrm{Diff}(M) \ltimes \Omega^2_{cl}(M)$

- preserve $\langle \cdot, \cdot \rangle$ on $TM \oplus T^*M$
- preserve $[\![ \cdot, \cdot ]\!]_H$

Diffeomorphisms: $\begin{pmatrix} \varphi_* & 0 \\ 0 & \varphi^{-1*} \end{pmatrix} \circlearrowright TM \oplus T^*M$ ; $B$-transform $e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$

$X + \xi \longmapsto X + \xi + \iota_X B$

Lie algebra $\cong TM \oplus T^*M$, acts by $[\![ X+\xi, \bullet ]\!]$ on $TM \oplus T^*M$

Automorphisms of $\mathcal{J}$: $\Phi^{-1} \mathcal{J} \Phi = \mathcal{J}$ for $\Phi \in \mathrm{Diff}(M) \ltimes \Omega^2_{cl}(M)$

Lie algebra = "generalized holomorphic vector fields" $\mathbb{X} \in \Gamma(T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M)$ s.t.

$d_L(\langle \mathbb{X}, \cdot \rangle|_L) = 0$ $\qquad$ ($L \subseteq T_{\mathbb{C}}M \oplus T^*M$, $\mathbb{C}$-Lie algebroid)

# GC branes (I)

<u>Defn (I)</u> (Gualtieri, Cavalcanti): A <u>GC brane</u> is a pair $(Y, \mathcal{F})$ of $Y \subseteq M$ and $\mathcal{F} \in \Omega^2(Y)$ s.t.

- $\iota_Y^* H = d\mathcal{F}$ $\quad (d\mathcal{F} = 0$ if $H = 0)$
- $J$ preserves $\tau_{\mathcal{F}} = \{X + \xi \in TY \oplus T^*M|_Y \;/\; \iota_Y^* \xi = \iota_X \mathcal{F}\}$

$B \in \Omega^2_{cl}(M)$

$e^{-B} J e^{B}$

$\tau_{\mathcal{F}} \mapsto e^{-B} \tau_{\mathcal{F}}$

$= \tau_{\mathcal{F} - \iota_Y^* B}$

<u>Note:</u> $\tau_{\mathcal{F}} \subseteq TM \oplus T^*M|_Y$ is max isotropic

$$0 \to \mathrm{Ann}(TY) \to \tau_{\mathcal{F}} \to TY \to 0 \quad \text{ses}$$

$\tau_{\mathcal{F}}$ = "generalized tangent bundle"

$$N(Y, \mathcal{F}) = {(TM \oplus T^*M)|_Y}\big/{\tau_{\mathcal{F}}}$$ "generalized normal bundle"

GC branes (II)

$\mathcal{l}$ is $(+i)$-eigenbundle of $\mathcal{J}$ in $\mathbb{T}_F \otimes \mathbb{C}$

Defn II (gualtieri) A GC brane is $(Y, F)$ as previously, together with an $\mathcal{l}$-representation, i.e. a complex v.s. (usually a line bundle) $E \to Y$ with a flat $\mathcal{l}$-connection:

$$\nabla : \Gamma(\mathcal{l}) \times \Gamma(E) \longrightarrow \Gamma(E)$$

st $[\nabla_\lambda, \nabla_{\lambda'}] = \nabla_{[\![\lambda, \lambda']\!]}$

Defn III. (Kapustin-Orlov, Koerber-Martucci, Collier) A GC brane is $(Y, F)$ as before together with a unitary line bundle $L \to Y$ with unitary connection $\nabla$, st $F_\nabla = F$

– prequantum line bundle, $[F] \in H^2(Y, \mathbb{Z})$

# String Theory

String theory: $X : \Sigma \longrightarrow M$

Riemann surface

spacetime mfd

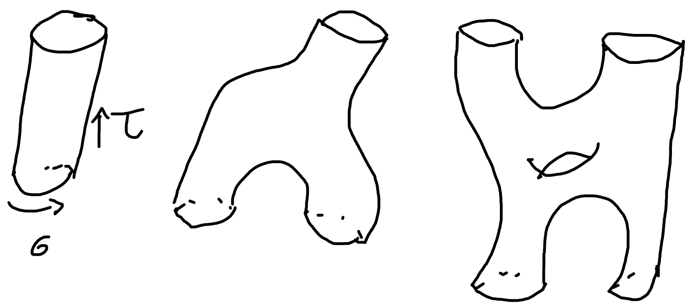
(Calabi-Yau)

Usually $M \cong \mathbb{R}^4 \times \hat{M}^6$

compact

Closed ST.

$\Sigma$ :

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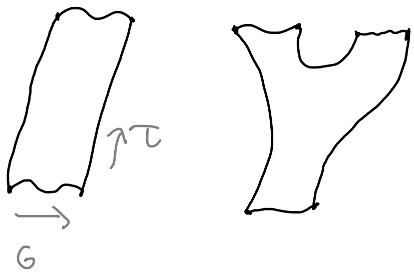
Branes in String Theory

Open string theory

$(\Sigma, \partial\Sigma)$

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Boundary conditions?

— von Neumann $\left( \frac{\partial X^i}{\partial \sigma}\Big|_{\partial\Sigma} = 0 \right)$

— Dirichlet $\left( X^j(\sigma)\Big|_{\partial\Sigma} = \text{const} \right)$

$\dim M = 2n$

$p$ von Neumann $\Longrightarrow$ string end moves inside $p$-dim. brane "D-branes"

$2n-p$ Dirichlet

Here: Top. A-model: Sympl mfd $(M, \omega)$. Branes: • Lag + flat hol l.b.

((2,2)-SUSY $\sigma$-model) + some coisotropics with hol. line bundle

GC branes in symplectic manifolds

Branes in top A-model = GC branes in symp. mfd
(Branes in top B-model = GC branes in cx mfds) $\Big\}$ Mirror symmetry

GC branes of $(M^{2n}, \omega)$ : $\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$

$(Y, F)$ st

- $Y$ coisotropic with presymp form $\omega_Y$

  Coisotropic: $TY^\omega \subseteq TY \iff \omega^{-1}(\mathrm{Ann}\, TY) \subseteq TY$

- $F$ presymp. form with $\ker(F) = \ker(\omega_Y)$ $\;\mathcal{E}$ !!

  $\iota_Y^* \omega = \omega_Y$

- on $TY/\mathcal{E}$. $\omega_Y^{-1} F =: I$ is cx str;

  ($\omega_Y$, $F$ symp.)

  $F + i\omega_Y$ hol. $(2,0)$-form

Possible dimensions: $n + 2k,\ k \in \mathbb{N}_0$ $\;(n, n+2, n+4, \ldots)$

$$\dim Y > \tfrac{1}{2} \dim M$$
"coisotropic A-branes"

Examples

(I) Lagrangian $\dim Y = \frac{1}{2} \dim M$

$$\iota_Y^* \omega = 0$$

$$F = 0$$

(II) Spacefilling $Y = M$

$\omega$ symp

$F$ symp.

$I = \omega^{-1} F$ is complex

$\Omega = F + i\omega$ is hol symp

# Examples (II)

(III) Mapping tori

$(N, \Omega)$ kd symp. , $\varphi \in \mathrm{Aut}(N, \Omega)$

$Y_\varphi := N \times \mathbb{R} \,/\, (x, t) \sim (\varphi(x), t + 2\pi)$

$\pi \quad Y_\varphi \longrightarrow S^1$

In part $\varphi = \mathrm{Id}$ , $Y = N \times S^1$

$M := Y \times \mathbb{R}_s = N \times S^1 \times \mathbb{R}_s$

$\omega = \mathrm{Im}(\Omega) + dt \wedge ds$

$E = \ker \omega_Y = \langle \frac{\partial}{\partial t} \rangle \simeq$

Leaf space of fol: $(N, \Omega)$

# Questions

What is the "space/category of coisotropic A-branes?"

(Fukaya category, mirror symmetry )

- When does a coisotropic submfd (in symp sense) admit brane structure?
- Given an A-brane, what A-branes are nearby?

  Given a coisotropic submfd near a brane, does it admit a brane structure?

$\rightsquigarrow$ small deformations of coisotropic brane

# Deformation theory

Deformation problem $\rightsquigarrow$ DGLA, MC elements $\quad (da + \frac{1}{2}[a,a) = 0\ )$

$L_\infty$-algebra? $(d, [\cdot,\cdot], l_3(\cdot,\cdot,\cdot), l_4(\cdot,\cdot,\cdot,\cdot), \dots)$

Small defos of coisotropic submfds: (symp Oh, Park, Poisson: Schätz, Zambon)

$E := \ker \omega_Y$

$NY \overset{\omega}{\cong} E^*$

Equip $E^*$ with str of $L_\infty$-algebroid ← Cattaneo, Felder

small defos of $Y$ $\overset{1:1}{\longleftrightarrow}$ Sections of $E^*$ which are Maurer-Cartan elements

Forgetful map: {deformation of coisotropic A-brane $(Y, F)$} $\longrightarrow$ {deformations of coisotr submfd}

Deformations of some coisotropic branes

[I] Lagrangian submfds (Exercise!)

$(Y, F) = (Y, 0)$

$L_\infty$-alg str on $\in^*$ trivial, only $d$

defos $\overset{1:1}{\longleftrightarrow}$ $\Omega^1_{cl}(Y)$

defos / Ham isotopy $\cong H^1(Y)$

brane defos = coisotropic defos

forgetful map = identity

(II) Spacefilling branes $M = Y \rightsquigarrow$ Hol symp. $(M, \Omega = F + i\omega)$

$Y$ stays the same, only $F$ can be deformed

forgetful map $\{\text{defos of } F\} \longrightarrow \{pt\}$

$M$ 4-mfd

$\ker \Omega = TM^{0,1}$

$\Omega \in \Omega^{2,0}(M)$, $[\Omega]$ det up to scaling

Local Torelli thm: $\{\text{small defos of } I\}$ $\overset{1:1}{\updownarrow}$ $\{\text{small defos of } [\Omega]\}$

Example and Counterexample

on $M = N \times S^1 \times \mathbb{R}$, $(N, \Omega)$ hol symp.

$Y = N \times S^1 \times \{0\}$

Coisotropic defor: Any $\{(p, s, f(n, s)) \mid p \in N, s \in S^1, f: Y \to \mathbb{R} \text{ smooth}\}$

But clearly not all of them are branes!

$\hookrightarrow$ More in Part II!