Decoupling Cartan connections and coupling G-structures: a Cartan bundle story

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The **curvature** of a Cartan connection θ is

$$\Theta = d\theta + \frac{1}{2}[\theta, \theta] \in \Omega^2(P, \mathfrak{g})$$

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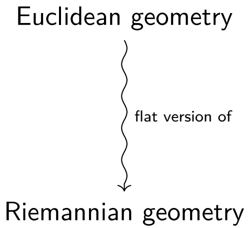
- Any flat Cartan geometry looks locally like a Klein geometry.

- Bernhard Riemann (1854): Riemannian manifolds locally modelled on Euclidean spaces



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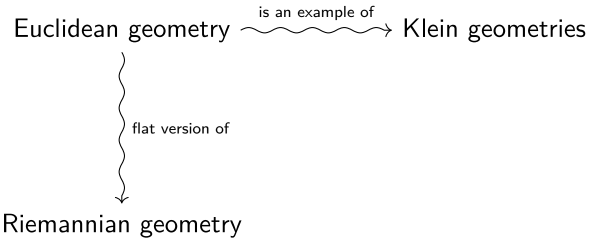

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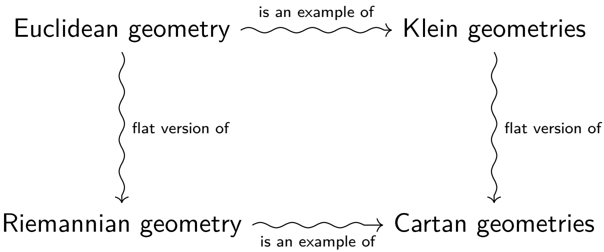
- Élie Cartan (1920's): *espaces généralisés* locally modelled on Klein geometries



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- **Conformal manifold** $=$ Cartan geometry modelled on S^n

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- $GL(n, \mathbb{C})$: Almost complex structures

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We also call $(P, \bar{\theta})$ an **abstract H-structure**

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- Abstract H-structures $\Leftrightarrow$ $\ker(\theta) = \ker(d\pi)$ and $V = \mathbb{R}^n$

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A **Lie groupoid** over M is a manifold $\mathcal{G}$ together with two surjective submersions $s, t : \mathcal{G} \to M$ and an associative partially defined multiplication $m : \mathcal{G}\ {}_s\times_t\ \mathcal{G} \to \mathcal{G}$ admitting units and inverses.

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How does our story look like from the groupoid point of view?

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Cartan bundles = Principal bundles + generalised Cartan connections	$\leftrightsquigarrow$	transitive Pfaffian groupoids = transitive Lie groupoids + "some (more general) kind of distribution"

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- (Morita) morphisms and equivalences between Cartan bundles
- Prolongations and towers of Cartan bundles
- Coefficients of Cartan bundles via (Atiyah) algebroids
- etc.

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Thank you for your attention

A short recap

- A Cartan bundle is a principal H-bundle $\pi : P \to M$ together with an H-representation V and a pointwise surjective H-invariant form $\theta \in \Omega^1(P, V)$, s.t. $\ker(\theta)$ is involutive and inside $\ker(d\pi)$

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Questions on the previous part?

Definition

A **multiplicative form** on a Lie groupoid $\mathcal{G} \rightrightarrows M$, with values in a representation $E \to M$ of $\mathcal{G}$, is a differential form $\omega \in \Omega^k(\mathcal{G}, t^*E)$ compatible with the multiplication m:

$$(m^*\omega)_{(g,h)} = g \cdot (pr_1^*\omega)_{(g,h)} + (pr_2^*\omega)_{(g,h)} \quad \forall (g,h) \in \mathcal{G} \,{}_s\!\times_t \mathcal{G}.$$

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Example

Multiplicative 0-forms are functions $f : \mathcal{G} \to \mathbb{R}$ such that

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Example

Symplectic groupoids are groupoids endowed with symplectic multiplicative 2-forms (with coefficient in $\mathbb{R}$).

Proposition (C.)

Let P be a principal H-bundle, V a H-representation and $\theta \in \Omega^1(P, V)$. The associated gauge groupoid $\mathcal{G}$ is endowed with

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Proof.

$$(H, 0) \overset{m_P}{\curvearrowright} (P, \theta)$$

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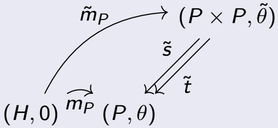

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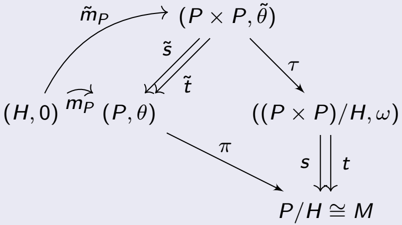

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The gauge groupoid of a Cartan geometry (P, θ) is a transitive Pfaffian groupoid $(\mathcal{G}, \omega)$ with zero symbol space.

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Example

The Pfaffian groupoid associated to a H-structure has symbol $\mathfrak{g}(\omega) = \ker(\rho)$.

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- Any other questions you might have (might not know the answer!)

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- $\mathfrak{l}$-connections ("partial connections")

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Proposition (Accornero, C.)

(P, θ) reductive Cartan bundle $\Rightarrow$ $pr_1 \circ \theta$ is a $\mathfrak{l}$-connection.

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(Morita) morphisms and equivalences

- (Weak) Cartan morphisms $\Leftrightarrow$ (weak) Pfaffian morphisms

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- Cartan model isomorphisms $\Leftrightarrow$ Pfaffian Morita equivalences
- Is curvature a Morita invariant?

Prolongations and towers of Cartan bundles

- $P \to M$ principal bundle $\Rightarrow J^1P \to M$ Cartan bundle
- (P, θ) Cartan bundle $\Rightarrow$ prolongation $P^{(1)} \subseteq J^1P$ Cartan bundle
- Cartan geometries are indefinitely prolongable ($P^{(1)} \to P$ is a bijection), in general there are obstructions related to $\mathfrak{g}(\omega)$
- P is flat $\Rightarrow P^{(1)}$ flat
- A tower of abstract prolongations has H-structures at the bottom, Cartan geometry at the top, and Cartan bundles between them

Coefficients via Pfaffian (Atiyah) algebroids

For any Pfaffian groupoid $(\mathcal{G}, \omega)$ over M, with $\omega \in \Omega^1(\mathcal{G}, t^*E)$ and Lie algebroid A

$$\begin{array}{ccccccc} \mathfrak{g}(\omega) & \hookrightarrow & A & \overset{\omega}{\twoheadrightarrow} & E & \xrightarrow{\cong} & A/\mathfrak{g}(\omega) \\ \big\downarrow & & \Big\| & & \big\downarrow & & \big\downarrow \\ \ker(\rho) & \hookrightarrow & A & \overset{\rho}{\twoheadrightarrow} & TM & \xrightarrow{\cong} & A/\ker(\rho) \end{array}$$

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- $\ker(\theta) = \ker(d\pi) \Rightarrow$ we recover $TM \cong P[V]$ (H-structures)