Junior Global Poisson Workshop
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Real forms of complexified Hamiltonian systems

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A first observation:

Let $\mathbb{C}^2$ with coordinates $z = x + iy$ and $w = a + ib$. The standard holomorphic symplectic form is $\Omega = dz \wedge dw$.

Complex dynamics on $\mathbb{C}^2$

$$H = z^2 w^2$$

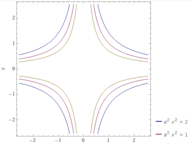
$$\downarrow$$

Real dynamics on $N = \mathbb{R}^2$

$$N_1 = \mathbb{R}^2 \text{ with } y = b = 0$$

$$\Omega|_{N_1} = dx \wedge da$$

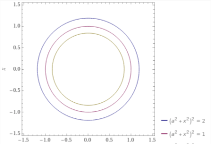
$$H|_{N_1} = x^2 a^2$$



$$N_2 = \mathbb{R}^2 \text{ with } x = b,\ y = a$$

$$\Omega|_{N_2} = 2dx \wedge da$$

$$H|_{N_2} = -(x^2 + a^2)^2$$



Complex world

Holomorphic symplectic geometry

Real form theory ↓ ↑ *Complexification*

Real world

Symplectic geometry

Dynamics

- Hamiltonian sytems
- Integrability

Geometry

- Poisson geometry/reduction
- Hyperkahler quotients

A compact real form of the spherical pendulum on $S^2 \times S^2$

Real forms

Let (M, I, Ω) holomorphic symplectic manifold with $\Omega = \omega_R + i\omega_I$

$\Rightarrow (M, \omega_R), (M, \omega_I)$ are both symplectic as real manifolds and

$$\omega_R(I(X), Y) = -\omega_I(X, Y) \quad \text{(Cauchy-Riemann).}$$

A **real form** N is a totally real submanifold of M of half dimension:

$$T_x N \oplus I(T_x N) = T_x M \quad \text{for each} \quad x \in M$$

- N is called a **real-symplectic form** if $\Omega|_N$ is purely real.
- N is called an **imaginary-symplectic form** if $\Omega|_N$ is purely imaginary.

N is a real-symplectic form of M $\Leftrightarrow$ N is a Lagrangian submanifold of (M, ω_I)

$\longrightarrow N$ is a symplectic submanifold of (M, ω_R).

Real structures

Let (M, Ω) be a holomorphic symplectic manifold.

A **real structure** is an anti-holomorphic involution $R : M \to M$

$$R^2 = Id \quad \text{and} \quad R_* \circ I = -I \circ R_*$$

$\longrightarrow$ The fixed point set M^R is a real form of M.

A real structure $R : M \to M$ is called

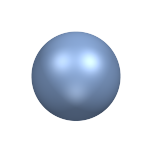
- a **real-symplectic structure** if $R^*\Omega = \bar{\Omega}$.
- an **imaginary-symplectic structure** if $R^*\Omega = -\bar{\Omega}$.

$\longrightarrow$ The fixed point sets M^R are real-symplectic and imaginary-symplectic forms of M, respectively.

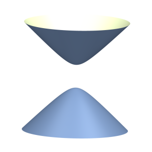
Examples

1) $(\mathbb{CS}^2, \Omega)$ is the affine variety $x^2 + y^2 + z^2 = 1$ in $\mathbb{C}^3$.

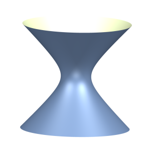
$R_1 : (x, y, z) \mapsto (\bar{x}, \bar{y}, \bar{z})$ $\quad R_2 : (x, y, z) \mapsto (\bar{x}, -\bar{y}, -\bar{z})$ $\quad R_3 : (x, y, z) \mapsto (\bar{x}, \bar{y}, -\bar{z})$



real-symplectic



real-symplectic



imaginary-symplectic

2) Let C be a complex manifold with real structure $r : C \to C$.
Cotangent lifts: $R : T^*C \to T^*C$

$$\langle R(\eta), X\rangle = \pm\overline{\langle \eta, r_* X\rangle}, \qquad \eta \in T^*_x C,\ X \in T_{r(x)}C.$$

$T^*C^r \simeq (T^*C)^R$ is either a real-symplectic form (+) or an imaginary-symplectic form (-) of (T^*C, Ω_{can}).

Real forms and dynamics

Let $H = u + iv$ be a holomorphic Hamiltonian on (M, Ω), $\Omega = \omega_R + i\omega_I$.

As in the real situation, the Hamiltonian vector field X_H is uniquely defined such that $\Omega(X_H, Y) = dH(Y)$ for each $Y \in T_x M$.

When can we define a "real form" of (M, Ω, H) ***?***

- Real "phase space": real-symplectic form $(N, \Omega|_N)$ of (M, Ω).
- Real dynamics: N should at least be invariant under the flow of X_H.

Theorem.

A real-symplectic form $N \subset M$ is invariant under the flow generated by $H = u + iv$ if and only if v is locally constant on N. In this case the flow on N is identical to the flow generated by $u|_N$ on $(N, \Omega|_N)$.

$\longrightarrow$ $(N, \Omega|_N, u|_N)$ *is a "real form" of* (M, Ω, H) *if* N *is a real-symplectic form of* (M, Ω) *which is invariant under the flow generated by* H.

Example

The complex 2-sphere $\mathbb{CS}^2$ is the affine variety $x^2 + y^2 + z^2 = 1$ in $\mathbb{C}^3$. We identify $T^*\mathbb{CS}^2 := \{(q,p) \in \mathbb{C}^6 \mid q^T q = 1,\ p^T q = 0\}$ with $\Omega_{can} = d\lambda$.

The real structures $r_1 : (x,y,z) \mapsto (\bar{x}, \bar{y}, \bar{z})$ and $r_2 : (x,y,z) \mapsto (\bar{x}, -\bar{y}, -\bar{z})$ on $\mathbb{CS}^2$ have fixed point sets

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad x^2 - y^2 - z^2 = 1$$

respectively (S^2  and $H^2 \sqcup H^2$ ).

$\longrightarrow$ yields real-symplectic structures R_1, R_2 on $T^*\mathbb{CS}^2$ with fixed point sets

$$N_1 = T^*S^2 \quad \text{and} \quad N_2 = T^*H^2$$

that are real-symplectic forms. The free holomorphic Hamiltonian $H : T^*\mathbb{CS}^2 \to \mathbb{C}$, $H(q,p) = p^T p$ is purely real on N_1 and N_2.

Real forms and integrability

Let (M^{2n}, Ω) be a connected holomorphic symplectic manifold.

A *holomorphic integrable system* on (M, Ω) is a collection of n holomorphic functions $F = (f_1, \ldots, f_n) : M \to \mathbb{C}^n$ such that

- $\{f_j, f_k\} = \Omega(X_{f_j}, X_{f_k}) = 0$ for each j, k,
- $df_1 \wedge \cdots \wedge df_n \neq 0$ on a dense subset of M.

Do we have a notion of "real form" for a holomorphic integrable system $F = (f_1, \ldots, f_n) : (M, \Omega) \to \mathbb{C}^n$ ***?***

Theorem.

Let $F = (f_1, \ldots, f_n) : (M, \Omega) \to \mathbb{C}^n$ be a holomorphic integrable system. If the flow generated by $H := f_1$ leaves an analytic, real-symplectic form $(N, \Omega|_N)$ invariant then the real Hamiltonian $H|_N$ is completely integrable on $(N, \Omega|_N)$.

Proposition.

Suppose $(N, \Omega|_N)$ is a connected, analytic real-symplectic form of (M, Ω) and

$$(u_1, \ldots, u_n) : (N, \Omega|_N) \to \mathbb{R}^n$$

is a real analytic integrable system. Then the holomorphic extensions of these functions in a neighbourhood $\mathcal{U} \subset M$ of N define a holomorphic integrable system.

$$
\begin{array}{ccc}
 & (M, \Omega) & \\
\swarrow & & \searrow \\
(N_1, \Omega|_{N_1}) & & (N_2, \Omega|_{N_2})
\end{array}
$$

The spherical pendulum

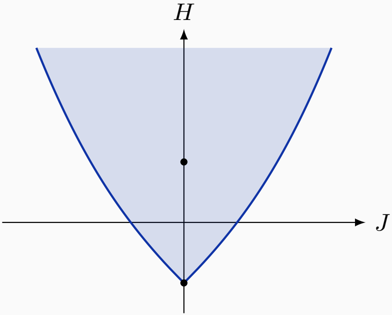
The Hamiltonian of the *spherical pendulum* $H : T^*S^2 \to \mathbb{R}$ is given by

$$H(q,p) = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) - q_3$$

with $p, q \in \mathbb{R}^3$ such that $q^T q = 1$ and $p^T q = 0$. It has the additional integral $J(q,p) = q_1 p_2 - q_2 p_1$ (angular momentum).

$$F = (J, H) : T^*S^2 \to \mathbb{R}^2$$

is a real integrable system with bifurcation diagram $F(T^*S^2)$



A compact real form of the pendulum

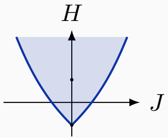
Complex system on
$(\mathbb{CS}^2 \times \mathbb{CS}^2, \Omega \oplus \Omega)$

$(S^2 \times S^2, \omega_{S^2} \oplus \omega_{S^2})$
This real form is
***real**-symplectic*

A new real system ?

(T^*S^2, ω_{can})
This real form is
***imaginary**-symplectic*

Spherical pendulum (H, J)



The tear drop

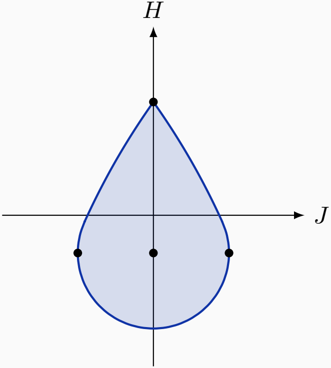
On $S^2 \times S^2$ with coordinates $(x_j, y_j, z_j) \in S^2 \subset \mathbb{R}^3$ and $j = 1, 2$, the new system has Hamiltonian

$$H = \frac{1}{2}(x_1x_2 + y_1y_2 - z_1z_2) + \frac{z_1 - z_2}{\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2 + (z_1 - z_2)^2}}$$

with additional integral $J = \frac{1}{2}(z_1 + z_2)$.

$$F = (J, H) : S^2 \times S^2 \to \mathbb{R}^2$$

is a real integrable system with bifurcation diagram $F(S^2 \times S^2)$



Thank you for your attention!