Non-algebraicity of hypercomplex nilmanifolds

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Overview of the talk

1. Plenary session
   - Complex manifolds
   - Nilmanifolds
   - Main results
2. Parallel session
   - Examples
   - Algebraic dimension
   - Subvarieties of hypercomplex nilmanifolds

Complex structures on manifolds

Let V be a vectorspace, $I\ 2\ \mathrm{End}(V)$, $I^2 = \quad 1$ an **almost complex structure.** Consider the eigenvalue decomposition

$$V \quad \mathbb{C} = V^{1;0} \quad V^{0;1}$$

$$Ix = p\overline{}1x \text{ for } x\ 2\ V^{1;0};\ Ix = \quad p\overline{}1x \text{ for } x\ 2\ V^{0;1}$$

Consider a smooth manifold X equipped with an **almost complex structure** $I\ 2\ \mathrm{End}(TX)$. Then one has the decomposition

$$TX \quad \mathbb{C} = T^{1;0}X \quad T^{0;1}X$$

Definition

An almost complex structure I on X is called **integrable** or just a **complex structure** if

$$[T^{1;0}X; T^{1;0}X] \quad T^{1;0}X$$

Newlander-Nirenberg theorem

Definition

A smooth map $f : X \quad ! \quad Y$ of almost complex manifolds is called holomorphic if $8x \ 2 \ X; v \ 2 \ T_x X$

$$D_x f(Iv) = I \quad D_x f(v)$$

Newlander-Nirenberg theorem

Let X be a smooth manifold with an almost complex structure I. Then I is integrable if and only if X is locally biholomorphic to an open ball in $\mathbb{C}^n$.

Remark

The integrability condition $[T^{1;0}X; T^{1;0}X] \quad T^{1;0}X$ is equivalent to the vanishing of the **Nijenhuis tensor** N

$$N(v;u) = [v;u] + I([v;Iu] + [Iv;u]) \quad [Iv;Iu] = 0 \quad 8 \text{ vector } \quad \text{elds } v;u$$

Kähler manifolds

Let V be a vector space with a complex structure I. Let g be an **Hermitian metric** on V i.e. a Euclidean metric on V s.t.

$$g(Iv;Iu) = g(v;u)$$

Then $!(v;u) := g(Iv;u)$ is a skew-symmetric 2-form. Let X be a complex manifold, g a Hermitian metric on X, $!(v;u) := g(Iv;u)$.

Definition

A complex manifold X is called **Kähler** if $d! = 0$.

Examples

Examples of Kähler manifolds

1. CP^n, all smooth projective varieties X $\quad CP^n$ (but not all Kähler ones are projective!);
2. Complex tori $\mathbb{C}^n = \ ;$
3. A complex submanifold of a Kähler manifold is Kähler.

Nilpotent Lie algebras and nilmanifolds

Let $\mathfrak{g}$ be a Lie algebra. Define $\mathfrak{g}_1 := [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}_i := [\mathfrak{g}, \mathfrak{g}_{i-1}]$. Then $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \ldots$ is called the **lower central series** of $\mathfrak{g}$.

Definition

A Lie algebra $\mathfrak{g}$ is called **nilpotent** if $\mathfrak{g}_k = 0$ for some k.

If k is the minimal number such that $\mathfrak{g}_k = 0$ then the Lie algebra $\mathfrak{g}$ is called k-**step nilpotent**.

Definition

Let G be a nilpotent Lie group and $\Gamma \subset G$ a cocompact lattice i.e. a discrete subgroup s.t. $\Gamma \backslash G$ is compact. Then $X := \Gamma \backslash G$ is called a **nilmanifold**.

Nota bene: in the definition of a nilmanifold we take the quotient by the **left** action of Γ. The group G acts on $X = \Gamma \backslash G$ **on the right**.

Complex structures on Lie groups

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra. Every $v \in \mathfrak{g}$ defines a **left-invariant vector field** $\tilde{v}$ on G. The map $v \mapsto \tilde{v}$ is an isomorphism of Lie algebras.

Let $L \in \mathrm{End}(\mathfrak{g})$ be an almost complex structure, $\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$. It induces a **left-invariant almost complex structure** $\tilde{L}$ on G.

Fact

The almost complex structure $\tilde{L}$ on G is integrable iff $\mathfrak{g}^{1,0}$ is **a Lie subalgebra of** $\mathfrak{g} \otimes \mathbb{C}$.

Proof: First, left-invariant vector fields on G generate the space of smooth vector fields on G over the smooth functions. Hence we can check the integrability condition just for them.

Complex nilmanifolds

Nota Bene

A left-invariant complex structure L on G makes G into a complex manifold **but in general not into a complex Lie group.** (An example is postponed until the parallel session)

A Lie group G is a complex Lie group i $\mathfrak{g}^{1;0}$ **is an ideal of** $\mathfrak{g}$ $\mathbb{C}$.

Definition

Let G be a nilpotent Lie group with a **left-invariant** complex structure L and G a cocompact lattice. Then $X :=$ nG is called a **complex nilmanifold**.

The **right** action of G on $X =$ nG need **not** preserve the complex structure.

Iwasawa manifold

The **complex Heisenberg group** of dimension 3 is

$$H = \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} ; \quad z_1, z_2, z_3 \in \mathbb{C} \right\}$$

An **Iwasawa manifold** is $\Gamma \backslash H$ where

$$\Gamma = \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} ; \quad z_1, z_2, z_3 \in \mathbb{Z}[\sqrt{-1}] \right\}$$

Remark

Iwasawa manifold is non-Kähler. Actually, **all complex nilmanifolds except of complex tori are non-Kähler.**

Hypercomplex manifolds

Notation: H is the quaternion algebra, it is generated by $I; J; K$,
$I^2 = J^2 = K^2 = \quad 1; IJ = \quad JI = K$.

Fact

An element L 2 H satis es $L^2 = \quad 1$ i $\quad L = xI + yJ + zK$, $x^2 + y^2 + z^2 = 1$.

Definition

A manifold X is called **almost hypercomplex** if H acts on TX. It is called **hypercomplex** if every complex structure on X induced from H is integrable.

Definition

Let G be a nilpotent Lie group with a **left-invariant** hypercomplex structure $(I; J; K)$ and $\quad G$ a cocompact lattice. Then $X := \quad nG$ is called a **hypercomplex nilmanifold**.

Main theorems: preliminary version

Notation: "$88L\ 2$ H" = "for all but a countable number of complex structures $L\ 2$ H."

Let X be a hypercomplex manifold. We denote by X_L the manifold X considered as a complex manifold with a complex structure $L\ 2$ H.

Theorem 1 (A.–Verbitsky)

Let X be a hypercomplex nilmanifold. Then $88L\ 2$ H the complex manifold X_L is does not admit a non-trivial meromorphic map onto a Kähler manifold.

Theorem 2 (A.–Verbitsky)

Let X be a hypercomplex nilmanifold **admitting an HKT-structure.** Then $88L\ 2$ H every complex subvariety of X_L is hypercomplex. In particular, every complex subvariety of X_L is even-dimensional.

Kodaira surface. Part 1

Define

$$G = \left\{ g(z_1, z_2) := \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset GL(3, \mathbb{C})$$

(z_1, z_2) are complex coordinates on G.
The left multiplication by $g(a_1, a_2)$ is given by

$$(z_1, z_2) \mapsto (z_1 + a_1, z_2 + a_1 z_1 + a_2) \quad \textbf{It's holomorphic}$$

The right multiplication by $g(a_1, a_2)$ is given by

$$(z_1, z_2) \mapsto (z_1 + a_1, z_2 + a_1 z_1 + a_2) \quad \textbf{It's not holomorphic!}$$

The group G is **not a complex Lie group** but admits a left-invariant complex structure.

Kodaira surface. Part 2

Define $\Gamma := GL(3, \mathbb{Z}[\sqrt{-1}]) \cap G$. Then the complex surface $X = \Gamma \backslash G$ is an example of a **Kodaira surface.** It is **not Kähler.** The map

$$\Gamma \backslash G = X \to E = \mathbb{C}/\mathbb{Z}[\sqrt{-1}] \quad (z_1, z_2) \mapsto z_1$$

is a principal elliptic fibration over the elliptic curve $E = \mathbb{C}/\mathbb{Z}[\sqrt{-1}]$.

Kodaira surface **does not** admit a hypercomplex structure.

"Doubling" construction. Part 1

Let X be a manifold equipped with a flat torsion-free affine connection $\nabla : TX \to TX \otimes \Lambda^1 X$.

$$[\nabla_v, \nabla_u] = \nabla_{[v,u]} \quad \text{(flat)}$$

$$\nabla_v u - \nabla_u v = [v,u] \quad \text{(torsion-free)}$$

Let $\pi : TX \to X$ denote the natural projection. ∇ induces the decomposition $T_x(TX) = H_x \oplus V_x$, $V_x := \ker d\pi$ for any point $x \in TX$.

$$V_x \cong H_x \cong T_{\pi(x)}X \implies T_x(TX) \cong T_{\pi(x)}X^{\oplus 2}$$

Define a complex structure J on a manifold TX as

$$J(v,u) := (-u, v) \quad (JV_x = H_x,\ JH_x = V_x)$$

Fact

In the assumptions above J is an integrable complex structure on a manifold TX

"Doubling" construction. Part 2

Assume that the monodromy of r preserves a lattice $_x$ T_xX. Then 9 a lattice TX parallel wrt r.

Fact

The manifold $TX=$ is a complex manifold. It is called a **"doubling"** of X.

Assume now that $(X; I)$ is a complex manifold and $rI = 0$. Then

$$T_x(\overline{T}X) = T_{(x)}X \quad \overline{T_{(x)}X}$$

and

$$I(v;u) := (Iv; \quad Iu) \quad J(v;u) := (\quad u; v) \quad K(v;u) := (\quad Iu; \quad Iv)$$

is an almost hypercomplex structure on the manifold $\overline{T}X$ (and $\overline{T}X=$ as well)

Fact

The constructed almost hypercomplex structure on $\overline{T}X$ is in fact hypercomplex.

"Doubling" construction. Part 3

Let's start with a Lie group G with a Lie algebra $\mathfrak{g}$. Left-invariant affine flat connections on G are in one-to-one correspondence with Lie-algebra representations

$$\nabla : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}) \quad v \mapsto \nabla_v$$

Assume also that ∇ is torsion-free.

We define a bracket on $T\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}$ as follows (the first $\mathfrak{g}$ is "horizontal", the second is "vertical")

$$[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2], \nabla_{x_1} y_2 - \nabla_{x_2} y_1)$$

and a complex structure J on $T\mathfrak{g}$ as $J(x, y) = (-y, x)$

Fact

The bracket $[\cdot, \cdot]$ makes $T\mathfrak{g}$ into a Lie algebra. The complex structure J is integrable. The hypercomplex analogue of this fact also holds.

How to measure non-algebraicity?

Let X be a compact complex manifold, $K(X)$ the eld of meromorphic functions on X

Definition

The **algebraic dimension** of X is the transcedence degree of $K(X)$.

Definition-Proposition

Consider a projective variety X^{alg} with a dominant rational map $r : X \; ! \; X^{alg}$. If $r \; : K(X^{alg}) \; ! \; K(X)$ is an isomorphism then X^{alg} is called an **algebraic reduction of** X. An algebraic reduction exists and is unique up to a birational isomorphism.

If X does not contain a divisor then it is of algebraic dimension zero. The opposite does not hold in general (though the opposite is true if X is assumed to be a complex torus).

Hyperkähler manifolds

Let X be a hypercomplex manifold. Let g be a **hyper-Hermitian metric** on X i.e. Hermitian wrt every complex structure $L \in \mathbb{H}$. Define $\omega_L(x, y) := g(Lx, y)$.

Definition

A hyper-Hermitian manifold X is called **hyperkähler** if $\forall L \in \mathbb{H} : d\omega_L = 0$. A hyper-Hermitian manifold X is called **HKT** if $\partial \Omega_I = 0$.

Examples

Examples of compact hyperkähler manifolds:

1. Hypercomplex tori $\mathbb{H}^n/\Gamma$;
2. K3-surfaces, their Hilbert schemes of points, etc

Non-example: A doubling of a non-Kähler complex manifold (f.e. Kodaira surface)

Hyperkähler manifolds are very non-algebraic

Theorem (Fujiki'87)

Let X be a compact hyperkähler manifold. Then $\forall L \in \mathbb{H}$ the complex manifold X_L is of algebraic dimension zero.

Definition

Let M be a hypercomplex manifold. A subvariety $M \subset X$ is called **trianalytic** if M is complex analytic wrt every complex structure $L \in \mathbb{H}$.

Theorem (Verbitsky'95)

Let X be a compact hyperkähler manifold. Then $\forall L \in \mathbb{H}$ every complex subvariety of X_L is trianalytic.

The second theorem implies the first one.

What about hypercomplex manifolds?

Theorems of Fujiki and Verbitsky do **not** hold for hypercomplex manifolds in general.

Examples

Let $X = \mathbb{H}^n/\lambda^{\mathbb{Z}}$, $\lambda \in \mathbb{R}_{>1}$. It is an example of a **Hopf manifold**. Then $\forall L \in \mathbb{H}$ there is an isotrivial elliptic fibration $X \to \mathbb{C}P^{2n-1}$, hence $\forall L \in \mathbb{H}$, X_L is of algebraic dimension $2n-1$ and contains an elliptic curve.

Definition-Proposition

Let X be a hypercomplex manifold. Then $\exists!$ torsion-free connection ∇ preserving the hypercomplex structure. It is called the **Obata connection**. If $Hol(\nabla)$ is contained in $SL(n, \mathbb{H})$ then X is called an $SL(n, \mathbb{H})$-**manifold**.

Theorem (Soldatenkov–Verbitsky'12)

Let X be an $SL(n, \mathbb{H})$-manifold admitting an HKT-metric. Then $\forall\forall X \in \mathbb{H}$ the manifold X_L does not contain divisors and every complex subvariety of X_L of codimension 2 is trianalytic.

Main theorems

We prove that the theorems of Fujiki, Verbitsky **do** hold (in some sense) for **hypercomplex nilmanifolds**

Theorem 1 (A.–Verbitsky)

Let X be a hypercomplex nilmanifold. Then $88L\ 2$ H the algebraic dimension of X_L is zero.

Theorem 2 (A.–Verbitsky), preliminary version

Let X be a hypercomplex nilmanifold **admitting an HKT-structure**. Then $88L\ 2$ H **every** complex subvariety of X_L is trianalytic.

Hypercomplex nilmanifolds are always $SL(n;$ H$)$-manifolds (Barberis{Dotti{Verbitsky'09).

Albanese variety

Let $X = \Gamma\backslash G$ be a complex nilmanifold. Then $\Lambda := \log(\Gamma)$ is a lattice in $\mathfrak{g}$ (Mal'cev'51). Consider the *minimal rational L-invariant subspace of* $\mathfrak{g}$ *containing* $[\mathfrak{g},\mathfrak{g}]$. Denote it by $[\mathfrak{g},\mathfrak{g}]_{\mathbb{Q},L}$. The quotient map $\mathfrak{g} \to \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]_{\mathbb{Q},L}$ induces a holomorphic map

$$r : \Gamma\backslash G = X \to T := (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]_{\mathbb{Q},L})/\Lambda$$

Definition

The torus T defined above is called the **Albanese variety** of a nilmanifold $X = \Gamma\backslash G$ and the map $r : X \to T$ is called the **Albanese map** of X.

Theorem (Fino–Grantcharov–Verbitsky'18)

Let $X = \Gamma\backslash G$ be a complex nilmanifold and T its Albanese variety. Then every meromorphic map from X to a Kähler manifold is uniquely factorized through the Albanese map $r : X \to T$.

The theorem implies that **algebraic dimensions of X and T coincide.**

Hypercomplex Albanese variety

Let now $X = \ nG$ be a hypercomplex nilmanifold. Consider the *minimal rational* H-*invariant subspace of* g *containing* $g_1 = [g; g]$. Denote it by $[g; g]_{Q;H}$. Similarly, we obtain a map

$$R : \ nG = X \quad ! \ T_H := (g=[g; g]_{Q;H})=$$

which preserves the hypercomplex structure.

Definition

The torus T_H de ned above is called the **hypercomplex Albanese variety** of a nilmanifold $X = \ nG$ and the map $R : X \quad ! \ T_H$ is called the **hypercomplex Albanese map** of X.

Lemma

Let $X = \ nG$ be a hypercomplex nilmanifold. Then *88L 2* H the hypercomplex Albanese map is the (complex) Albanese map of X_L.

Hypercomplex Albanese vs Complex Albanese

Lemma

Let $X = \ nG$ be a hypercomplex nilmanifold. Then $88L\ 2$ H the hypercomplex Albanese map is the (complex) Albanese map of X_L.

Proof:

Observation

Let V be an H-vector space with a rational structure. Then $88L\ 2$ H every rational L-invariant space is H-invariant.

Indeed, if an L-invariant space is invariant wrt L^0 6= L then it is H-invariant. Hence the set of complex structures $L\ 2$ H s.t. there exist an L- but not H-invariant rational subspace of V is countable.

By applying the observation to $V =$ g we obtain that $88L\ 2$ H: $[g; g]_{Q;L} = [g; g]_{Q;H}$.

Proof of the First theorem

Theorem 1 (A.–Verbitsky)

Let X be a hypercomplex nilmanifold. Then $88L\ 2$ H the algebraic dimension of X_L is zero.

Proof. Let T be the hypercomplex Albanese variety. We saw in the previous slides that $88L\ 2$ H we have

$$\operatorname{alg\,dim} X_L = \operatorname{alg\,dim} T_L$$

The torus T is hyperkähler, hence $88L\ 2$ H the algebraic dimension of T_L is zero.

Abelian complex structures

Let $\mathfrak{g}$ be a Lie algebra with a complex structure L.

Definition

The complex structure L is called **abelian** if $\mathfrak{g}^{1,0}$ is an abelian subalgebra of $\mathfrak{g} \otimes \mathbb{C}$. Equivalently,

$$\forall x, y \in \mathfrak{g} : [Lx, y] = -[x, Ly]$$

Suppose that $\mathfrak{g}$ admits a hypercomplex structure (I, J, K). Then J, K are abelian whenever the complex structure I is abelian (Dotti–Fino'03). If one (hence any) complex structure $L \in \mathbb{H}$ is abelian then the hypercomplex structure on $\mathfrak{g}$ is called **abelian**.

Theorem (Dotti–Fino'01, Barberis–Dotti–Verbitsky'09, also Fino–Grantcharov'03)

Let X be a hypercomplex nilmanifold. Then X admits an HKT-metric iff the hypercomplex structure is abelian

Locally homogeneous submanifolds

Let G be a Lie group. We trivialize TG by **left** multiplications. If G is a Lie group with a **left-invariant** complex structure which is not right-invariant then this trivialization is complex but **not holomorphic** because

Nota bene

The ow of a **left-invariant** vector eld is the multiplication **on the right** by exp( ).

This trivialization of TG descends to a, generally speaking, **non-holomorphic** complex trivialization of TX where $X = nG$.

Definition

A submanifold M X is called **locally homogeneous** if $8x\ 2\ M$ the tangent space T_xM is identi ed with a xed subspace h g via the trivialization of TX above.

The subspace h g is automatically a rational subalgebra.

Second theorem. Step 1: case of tori

Theorem 2 (A.–Verbitsky): Final version

Let $X = \ nG$ be an abelian hypercomplex nilmanifold. Then $88L\ 2$ H every complex subvariety of X_L is a trianalytic locally homogeneous submanifold of X.

Sketch of the proof. Step 1. The claim is known to hold for a hypercomplex torus T. Indeed, $88L\ 2$ H every complex subvariety of T_L is trianalytic. A trianalytic subvariety of a hyperkahler manifold is totally geodesic (Verbitsky'96). Hence every trianalytic subvariety of T is a subtorus.

Step 2: Principal toric fibration

Lemma

Let g be a Lie algebra with an abelian Lie structure L. Then its center z is L-invariant.

Proposition

Let $X = \ \ nG$ be an abelian complex nilmanifold. Let Z denote the center of G. Then the map

$$: \ \ nG = X \quad ! \ \ Y := \ \ nG{=}Z$$

is a holomorphic principal toric bration with a ber $T = Z{=}(\ \ \backslash Z)$.

Proof: The right action of Z on G is holomorphic because it coincides with the right action. Hence the right action of $Z{=}(\ \ \backslash Z)$ on $X = \ \ nG$ is also holomorphic.

Step 3: Induction step. The reduction to the case of a multisection

Let $M \quad X_L$ be a complex subvariety. Consider the principal bration

$$: \quad nG = X \quad ! \; Y := \quad nG{=}Z$$

By induction hypothesis both (M) **and the fibers of** j_M **are trianalytic locally homogeneous submanifolds.** One can use this observation to show that

Fact

It is actually enough to assume that M is a multisection of $: X \quad ! \quad Y$ i.e. the map $j_M \colon M \quad ! \quad Y$ is surjective and generically nite.

Step 4: Multisections are étale

Consider the principal fibration $: nG = X \to Y := nG=Z$. Let M X_L be a multisection of . Consider the Stein factorization of the map $j_M : Y$

$$M \xrightarrow{1} Y' \xrightarrow{2} Y$$

The map $_1$ is a **birational transformation** and the map $_2$ is **finite**.

Observation 1

The **branch locus** of $_2$ is a divisor in $Y \Rightarrow$ the map $_2$ is **étale** (Y has no divisors by the induction hypothesis).

Observation 2

The **exceptional locus** E M of $_1$ is a divisor in $M \Rightarrow$ if non-empty, E has odd dimension $\Rightarrow$ the map $: M \to Y$ has an odd-dimensional fiber. But $\forall y \in Y$ all the subvarieties of $^1(y)$ are trianalytic.

Hence $j_M : M \to Y$ is finite étale.

The end of the proof

Consider the principal T-fiber bundle $\pi : \Gamma\backslash G = X \to Y := \Gamma\backslash G/Z$. Define $T_k := T/\{k\text{-torsion}\}$. Consider the associated principal T_k-bundle $X_k := X \times T_k/T \to Y$. The manifold X_k is a nilmanifold as well.

Observation 3

A multisection $M \subset X_L$ of degree k gives rise to a section of $X_k \to Y$. Hence $X_k = Y \times T_k$

Observation 4

By [Maltsev'51] any decomposition $X_k = Y \times T_k$ comes from a Lie algebra decomposition $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{h}$. Here $\mathfrak{z}$ is the center of $\mathfrak{g}$. The existence of such a decomposition contradicts the nilpotency assumption on $\mathfrak{g}$.