Equivariant Cohomology Models for Differentiable Stacks

Luis Alejandro Barbosa-Torres

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Our goal

What we have

1. The concept of an action by a Lie group on a stack $\mathcal{M}$ and the concept of equivariant morphism.
2. The quotient stack $\mathcal{M}/G$.

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What we want

1. Find an atlas, $Y \to \mathcal{M}/G$, for the quotient stack $\mathcal{M}/G$.
2. Describe the Lie groupoid $(Y \times_{\mathcal{M}/G} Y \rightrightarrows Y)$ associated to $\mathcal{M}/G$.
3. Compute the homotopy type of $\mathcal{M}/G$,that is,

$$\| \, \text{Nerve} \, (Y \times_{\mathcal{M}/G} Y \rightrightarrows Y) \|.$$

2-Yoneda Lemma

Theorem

Let $\mathcal{M}$ be a stack over Diff and T a smooth manifold. There is an equivalence of categories

$$\mathrm{Hom}_{\mathsf{Stacks}}(T, \mathcal{M}) \cong \mathcal{M}(T).$$

Differentiable G-Stack

Definition

A G-stack $\mathcal{M}$ is called a *differentiable G-stack* if there is a smooth manifold X with an action by G and an equivariant morphism of G-stacks $p : X \to \mathcal{M}$ such that p is an atlas. The map $X \to \mathcal{M}$ is then called a G-atlas of $\mathcal{M}$.

Differentiable G-Stack

There exists a canonical map $q : \mathcal{M} \to \mathcal{M}/G$ induces by trivial principal G-bundles.

Proposition

The map $q : \mathcal{M} \to \mathcal{M}/G$ has local sections.

Proof

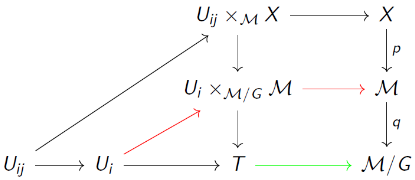
$$\begin{array}{ccccc} G \times U_i & \longrightarrow & E & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow & & \\ U_i & \longrightarrow & T & & \end{array}$$

Differentiable G-Stack

Proposition

Let $X \xrightarrow{p} \mathcal{M}$ be a G-atlas. Then we get an atlas for the quotient stack given by $X \xrightarrow{p} \mathcal{M} \xrightarrow{q} \mathcal{M}/G$.

Proof



Cohomology of Quotient Stacks

Homotopy type

The homotopy type for the quotient stack is given by the fat geometric realisation of the Lie groupoid $(X \times_{\mathcal{M}/G} X \rightrightarrows X)$

Proposition

We have the equivalences of stacks

$$X \times_{\mathcal{M}/G} X \cong (G \times X) \times_{\mathcal{M}} X \cong G \times (X \times_{\mathcal{M}} X)$$

For this, we consider the 2-commutative diagram

$$\begin{array}{ccccc}
E & \longrightarrow & G \times X & \xrightarrow{\sigma} & X \\
\downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle id_G \times p} & & \downarrow{\scriptstyle p} \\
G \times X & \xrightarrow{id_G \times p} & G \times \mathcal{M} & \xrightarrow{\mu} & \mathcal{M} \\
\downarrow{\scriptstyle Pr_2} & & \downarrow{\scriptstyle Pr_2} & & \downarrow{\scriptstyle q} \\
X & \xrightarrow{p} & \mathcal{M} & \xrightarrow{q} & \mathcal{M}/G
\end{array}$$

The Lie groupoid structure

How to find them

We have the following maps

$$(x, g\cdot y : q\circ p(x) \to q\circ p(g\cdot y)) \qquad X \times_{\mathcal{M}/G} X \longrightarrow X$$

$$\uparrow \cong$$

$$((g,x), g\cdot y; g\cdot p(x) \to p(g\cdot y)) \qquad (G\times X)\times_{\mathcal{M}} X$$

$$\uparrow \cong$$

$$(g,(x,y;p(x)\to p(y))) \qquad G\times (X\times_{\mathcal{M}} X)$$

The Nerve

How to find them

In general, we get that for a $n+1$-product

$$X \times_{\mathcal{M}/G} X \times_{\mathcal{M}/G} \dots \times_{\mathcal{M}/G} X \cong G^n \times X_n.$$

Using the same idea as before we get the faces.

Faces of the Nerve

For $T \in \mathrm{Diff}$,

$$\partial_i : G^{n+1} \times X_{n+1}(T) \to G^n \times X_n(T)$$

such that

$$\partial_i(g_1, g_2, \dots, g_{n+1}, (x_1, x_2, \dots, x_{n+2}; p(x_1) \to \dots \to p(x_{n+2})))$$

is equal to

$$(g_2, g_3, \dots, g_{n+1}, \pi_1(x_1, x_2, \dots, x_{n+2}; p(x_1) \to \dots \to p(x_{n+2})))$$

$$(g_1, \dots, g_i \cdot g_{i+1}, \dots, g_{n+2}, \pi_i(x_1, x_2, \dots, x_{n+2}; p(x_1) \to \dots \to p(x_{n+2})))$$

$$(g_1, g_2, \dots, g_n, g_{n+1} \cdot \pi_{n+2}(x_1, x_2, \dots, x_{n+2}; p(x_1) \to \dots \to p(x_{n+2})))$$

Simplicial action on the nerve

If we consider the fibered n-product $X_n = X \times_{\mathcal{M}} \ldots \times_{\mathcal{M}} X$, we define

$$\sigma_{n,T} : G \times X_n(T) \to X_n(T)$$

such that

$$\sigma_{n,T}(g, (x_1, x_2, \ldots, x_n; p(x_1) \to \ldots \to p(x_n)))$$

$$= (g \cdot x_1, g \cdot x_2, \ldots, g \cdot x_n; p(g \cdot x_1) \to \ldots \to p(g \cdot x_n))$$

this morphism is well-defined since there exists
$g \cdot p(x_1) \to \ldots \to g \cdot p(x_n)$ and $g \cdot p(z) \cong p(g \cdot z)$ for any $z \in X(T)$.

One step up

Theorem

For a bisimplicial smooth manifold $X_{\bullet,\bullet}$, we have that there are homeomorphisms such that:

$$\| X_{\bullet,\bullet} \| \cong \| p \mapsto X_{p,p} \| \cong \| p \mapsto \| q \mapsto X_{p,q} \| \| \cong \| q \mapsto \| p \mapsto X_{p,q} \| \| .$$

Then we can consider the bisimplicial smooth manifold $\{G^\bullet \times X_\bullet\}$ given by $G^p \times X_q$ with horizontal faces given by the previous ones and vertical ones by the induced ones by $X_\bullet$. Then we notice that diagonal of this bisimplicial smooth manifold coincides with nerve we describe before.

If we consider the bisimplicial smooth manifold $\{G^\bullet \times X_\bullet\}$, then

Theorem

$$H^*(\mathcal{M}/G, \mathbb{R}) \cong H^*(EG \times_G \| X_\bullet \|, \mathbb{R})$$

with $\| X_\bullet \|$ *the fat geometric realisation of the stack* $\mathcal{M}$*.*

Classical Case

What happen if $\mathcal{M} = X$ a smooth manifold?

Definition

Let G be a Lie group and $\mathcal{M}$ a differentiable G-stack with a G-atlas $X \xrightarrow{p} M$. The *equivariant cohomology* of $\mathcal{M}$, $H^*_G(\mathcal{M}, \mathbb{R})$, is given by

$$H^*_G(\mathcal{M}, \mathbb{R}) = H^*(\mathcal{M}/G, \mathbb{R}).$$

Cartan Model

Complex of Equivariant Forms.

Let $X_\bullet$ be a simplicial smooth manifold and G a compact Lie group acting on $X_\bullet$. Consider the complex $C^\bullet$ given by

$$C^r = \bigoplus_{r=n+q} \Omega^q_G(X_n)$$

with differential $d_G + (-1)^q \partial$, where d_G is the Cartan differential and ∂ is the differential associated to the simplicial structure of $X_\bullet$.

Cohomology of Cartan Model

Proposition

If G is a connected compact Lie group, we have that

$$H^*_G(X_\bullet) \cong H^*(EG \times_G \| X_\bullet \|, \mathbb{R})$$

with $\| X_\bullet \|$ the fat geometric realisation of the simplicial manifold $X_\bullet$ constructed from the atlas.

Cohomology of Cartan Model

Proof

Consider the double complexes $F^{\bullet,\bullet}$ and $C^{\bullet,\bullet}$

$$F^{s,n} = \bigoplus_{q+p=s} \Omega^q(G^p \times X_n)$$

and

$$C^{s,n} = \Omega^s_G(X_n)$$

Then we can consider the spectral sequences given by $E_0^{s,n} = F^{s,n}$ and

Cohomology of Cartan Model

Proof

Then we have the following spectral sequences:

$$E_0^{s,n} = F^{s,n} \longrightarrow E_0'^{s,n} = C^{s,n}$$

$$E_1^{s,n} = H_G^s(X_n, \mathbb{R}) \xrightarrow{\cong} E_1'^{s,n} = H_G^s(X_n)$$

$$E_\infty^{s,n} = H_G^{s+n}(\Omega^*(G^\bullet \times X_\bullet)) \xrightarrow{\cong} E_\infty'^{s,n} = H^{s+n}(C^\bullet)$$

Cohomology of Cartan Model

Theorem

Let $\mathcal{M}$ be a differentiable G-stack with a G-atlas $X \to \mathcal{M}$ with G a connected compact Lie group. Then

$$H_G^*(\mathcal{M}) \cong H_G^*(X_\bullet).$$

Thank you

¡Gracias!