Equivariant Cohomology Models for Differentiable Stacks

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Classical Problem

Let G be a Lie group and M a smooth manifold with an action of G on M.

Question

How can I get a good notion of cohomology for M/G even when the action of G is not free?

Borel Model

Idea

Try to find a contractible space E with a free G-action and compute the cohomology of $(E \times M)/G = E \times_G M$ where the action of G is free on the product.

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Definition

We can define the *equivariant cohomology* as

$$H^*_G(M) = H^*(E \times_G M).$$

Transformation groupoid

Let G be a Lie group acting on a smooth manifold X.

Structure

The *transformation groupoid* $(G \times X \rightrightarrows X)$ is a Lie groupoid with source map as the projection on X and target map is the action by G on X.

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Nerve

The nerve is given by

$$\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; G^2 \times X \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; G \times X \;\rightrightarrows\; X$$

Faces

Faces of the nerve

The i-th face of the nerve is given by

$$\partial^i : G^n \times X \to G^{n-1} \times X$$

$$\partial^0(g_1, g_2, \ldots, g_n, x) = (g_2, \ldots, g_n, x),$$

$$\partial^i(g_1, g_2, \ldots, g_n, x) = (g_1, \ldots, g_i g_{i+1}, \ldots, g_n, x) \text{ for } 0 < i < n$$

$$\partial^n(g_1, g_2, \ldots, g_n, x) = (g_1, \ldots, g_{n-1}, g_n x)$$

Its fat geometric realisation

Small recall

Let $X_\bullet$ be a simplicial smooth manifold then its *fat geometric realisation* is the quotient space

$$\|X_\bullet\| = \|p \mapsto X_p\| = \bigcup_{p \in \mathbb{N}} \Delta^p \times X_p / \sim$$

with the identifications $(\partial^i t, x) \sim (t, \partial_i x)$, for any $x \in X_p$, $t \in \Delta^{p-1}, i, j = 0, \ldots, n$ and p.

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In our example

$$\|\mathrm{Nerve}(G \times M \rightrightarrows M)\| \cong EG \times M/G = EG \times_G M$$

Cartan Model

Let G be compact Lie group acting on the smooth manifold M.

Equivariant forms

These *equivariant forms* are defined as polynomial maps $\alpha : \mathfrak{g} \to \Omega^*(M)$, where $\mathfrak{g}$ is the Lie algebra of G and $\Omega^*(M)$ is the De Rham complex of differential forms such that

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\alpha} & \Omega^*(M) \\ \downarrow{\scriptstyle \mathrm{Ad}_g} & & \downarrow{\scriptstyle g} \\ \mathfrak{g} & \xrightarrow{\alpha} & \Omega^*(M) \end{array}$$

with Ad_g, the adjoint action by $g \in G$.

Complex of Equivariant Forms (Cartan Model)

The space of equivariant n-forms can be expressed as the invariant forms:

$$\Omega^n_G(M) = \bigoplus_{2k+i=n} (S^k(\mathfrak{g}^\wedge) \otimes \Omega^i(M))^G.$$

This forms a cochain complex with differential given by $d_G = d_{dR} - \iota$ where d_{dR} is the exterior derivative and ι is interior product.

Theorem (Cartan 1950)

If G is a compact Lie group acting on a smooth compact manifold then the complex of equivariant forms computes the equivariant cohomology in the Borel model.

Stacks

We consider Diff as the big site of local diffeomorphisms on the category of smooth manifolds and smooth maps.

Definition

A *stack* $\mathcal{M}$ over Diff is a pseudo-functor

$$\mathcal{M} : \mathsf{Diff}^{op} \to \mathsf{Grpds} \subset \mathsf{Cat}$$

such that:

1. We can glue objects.
2. We can glue morphisms.

Stack

Examples

- For any smooth manifold $X \in \mathrm{Diff}$ we can associate a stack given by

$$\underline{X} = \mathrm{Map}(\_, X) : \mathrm{Diff}^{op} \to \mathrm{Grpds}$$

which takes a $Y \in \mathrm{Diff}$ and associates the set of all smooth morphisms between Y and X, $\mathrm{Map}(Y, X)$. The morphisms in $\mathrm{Map}(Y, X)$ as a groupoid, are identity maps.

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- Let G be a Lie group. Consider the functor $\mathcal{B}G : \text{Diff}^{op} \to \text{Grpds}$ which assigns to any smooth manifold M, the category of principal G-bundles over M.

Examples

Examples

- Let G be a Lie group acting on a smooth manifold X. Then the *quotient stack* $[X/G]$ is defined by

$$[X/G] : \mathrm{Diff}^{op} \to \mathrm{Grpds}$$

$$Y \mapsto [X/G](Y) =$$

$$\langle (P \xrightarrow{p} Y, P \xrightarrow{f} X) \mid p \text{ a } G\text{-bundle, } f \text{ is } G\text{-equivariant} \rangle$$

with morphism given by $\phi : P \to P'$ such that $p' \circ \phi = p$ and $f' \circ \phi = f$ with ϕ a G-equivariant morphism.

Differentiable stack

Definition

A stack $\mathcal{M}$ is called a *differentiable stack* if there is a smooth manifold X and a morphism of stacks $p : X \to \mathcal{M}$ between the stack associated to X and the stack $\mathcal{M}$ such that:

1. For all morphisms of stacks $Y \to \mathcal{M}$, where Y is a smooth manifold, the stack associated to $X \times_{\mathcal{M}} Y$ is isomorphic to a smooth manifold.
2. p is a submersion, i.e., for all $Y \to \mathcal{M}$ the projection $X \times_{\mathcal{M}} Y \to Y$ is a submersion.

The map $X \to \mathcal{M}$ is then called an *atlas* of $\mathcal{M}$.

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With this atlas we can consider the Lie groupoid $X \times_{\mathcal{M}} X \rightrightarrows X$. This groupoid is called the *Lie groupoid associated to the differentiable stack* $\mathcal{M}$.

Lie groupoids and Differentiable stacks

Relation between these two categories

1. From any Lie groupoid, we can define a differentiable stack.
2. Any two Lie groupoids define the same differentiable stack if there exists a weak equivalence (Morita equivalence) between them.

Cohomology

Let $\mathcal{M}$ be a differentiable stack with atlas $X \to \mathcal{M}$ and consider the De Rham complex of differential forms Ω^* with the double complex $(\Omega^n_{dR}(X_k), d, \partial)$ where d is the exterior derivative and ∂ is the differential that comes from the alternate sum of the pullbacks of the faces in $X_\bullet$.

De Rham Cohomology

The cohomology given by the complex $(\Omega^\bullet_{dR}(X_\bullet), d, \partial)$ is called the *de Rham cohomology* of $\mathcal{M}$, and it is denoted by $H^n_{dR}(\mathcal{M})$.

Cohomology

Homotopy type

The homotopy type of the differential stack $X \to \mathcal{M}$ is given by the homotopy type of the fat geometric realisation $\| X_\bullet \|$.

Action on a Stack

Let G be a Lie group and $\mathcal{M}$ a stack.

Definition [Romagny 2005, Ginot-Noohi 2018]

An *action* of G on the stack $\mathcal{M}$ is given by the following data: A morphism $\mu : G \times \mathcal{M} \to \mathcal{M}$ such that for $T \in Diff$ we have $\mu_T : G \times \mathcal{M}(T) \to \mathcal{M}(T)$ with $(g, x) \mapsto g \cdot x$. And the following diagrams

$$\begin{array}{ccc} G \times G \times \mathcal{M} & \xrightarrow{m \times id_{\mathcal{M}}} & G \times \mathcal{M} \\ {\scriptstyle id_G \times \mu} \downarrow & \overset{\alpha}{\nearrow\!\!\!\nearrow} & \downarrow {\scriptstyle \mu} \\ G \times \mathcal{M} & \xrightarrow{\quad \mu \quad} & \mathcal{M} \end{array}$$

Action On A Stack

and

$$\begin{array}{ccc} G\times\mathcal{M} & \xrightarrow{\mu} & \mathcal{M} \\ {\scriptstyle e\times id_{\mathcal{M}}}\uparrow & \overset{\Leftarrow}{\scriptstyle \alpha} & \nearrow {\scriptstyle id_{\mathcal{M}}} \\ \mathcal{M} & & \end{array}$$

are 2-commutative. We say that the stack $\mathcal{M}$ is a G-stack if G acts on $\mathcal{M}$.

Equivariant morphism

Definition

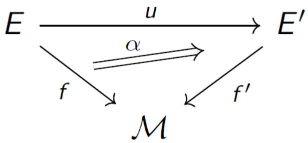
A *morphism of G-stacks* is a morphism of stacks $F : \mathcal{M} \to \mathcal{N}$ together with a 2-morphism σ with the following 2-commutative diagram

$$\begin{array}{ccc} G \times \mathcal{M} & \xrightarrow{\mu} & \mathcal{M} \\ {\scriptstyle id_G \times f}\big\downarrow & \overset{\sigma}{\nearrow\!\!\!\nearrow} & \big\downarrow{\scriptstyle f} \\ G \times \mathcal{N} & \xrightarrow{\nu} & \mathcal{N} \end{array}$$

Quotient Stack

Definition

We define the *quotient stack* $\mathcal{M}/G$ with objects in $\mathcal{M}/G(T)$ being $t = (p, f)$ such that $p : E \to T$ is a principal G-bundle and equivariant morphism $f : E \to \mathcal{M}$. As isomorphism between t and t', we have pairs (u, α) with a G-morphism $u : E \to E'$ and a 2-commutative diagram of G-stacks.



What was the goal at this point?

Idea

1. Find an atlas, $Y \to \mathcal{M}/G$, for the quotient stack $\mathcal{M}/G$.
2. Describe the Lie groupoid $(Y \times_{\mathcal{M}/G} Y \rightrightarrows Y)$ associated to $\mathcal{M}/G$.
3. Compute the homotopy type of $\mathcal{M}/G$,that is,

$$\| \text{ Nerve } (Y \times_{\mathcal{M}/G} Y \rightrightarrows Y)\|.$$

What we got

Theorem

For a G-atlas $X \to \mathcal{M}$, we have:

$$H^*(\mathcal{M}/G, \mathbb{R}) \cong H^*(EG \times_G \| X_\bullet \|, \mathbb{R})$$

with $\| X_\bullet \|$ the fat geometric realisation of the stack $\mathcal{M}$.

Remark

A G-atlas $X \to \mathcal{M}$ is an atlas between two G-stacks that preserves the action.

Definition

Let G be a Lie group and $\mathcal{M}$ a differentiable G-stack with a G-atlas $X \xrightarrow{p} M$. The *equivariant cohomology* of $\mathcal{M}$, $H^*_G(\mathcal{M}, \mathbb{R})$, is given by

$$H^*_G(\mathcal{M}, \mathbb{R}) = H^*(\mathcal{M}/G, \mathbb{R}).$$

Is there any other model?

Yes!

Simplicial Cartan Model [Meinreken 2003]

Let $X_\bullet$ be a simplicial smooth manifold and G a compact Lie group acting on $X_\bullet$. Consider the complex $C^\bullet$ given by

$$C^p = \bigoplus_{p=r+n} \Omega^r_G(X_n)$$

with the differential $d_G + (-1)^r\partial$, where d_G is the Cartan differential and ∂ is the differential that comes from the simplicial structure of $X_\bullet$.

What do we get?

Theorem

Let $\mathcal{M}$ be a differentiable G-stack with a G-atlas $X \to \mathcal{M}$ with G a compact Lie group. Then

$$H^*_G(\mathcal{M}) \cong H^*_G(X_\bullet).$$

Ongoing work

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As a joint work with Frank Neumann (Leicester)

Let $\mathcal{M}$ be a differentiable G-stack. Then we have:

- The isomorphism classes of G-equivariant principal $\mathbb{T}$-bundles over $\mathcal{M}$ are classified by $H^1_G(\mathcal{M}, \mathbb{T})$.
- The isomorphism classes of G-equivariant gerbes with band $\mathbb{T}$ over $\mathcal{M}$ are classified by $H^2_G(\mathcal{M}, \mathbb{T})$.

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As joint work with Cristian Ortiz-Ph.D. Student James Simeao (Sao Paulo)

Try to conceive an equivariant cohomology of 2-group actions.

Thank you!

¡Gracias!