Invariant Generalized Complex Structures on Flag Manifolds

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Supported by FAPESP 2020/12018-5

JUNIOR GLOBAL POISSON WORKSHOP II

May 5, 2021

Summary

Introduction

- The notion of generalized complex structure was introduced by N. Hitchin, in the article *Generalized Calabi–Yau manifolds*, and initially developed by M. Gualtieri and G. Cavalcanti.
- It allows us to study complex and symplectic geometry in a unique framework.
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Generalized complex geometry

Let V be an n-dimensional vector space and let V^* be its dual space. We can define a symmetric bilinear form on $V \oplus V^*$ by

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\xi(Y) + \eta(X)),$$

where $X, Y \in V$ and $\xi, \eta \in V^*$. Such bilinear form is nondegenerate and has signature (n, n).

Definition

A (linear) generalized complex structure on V is an endomorphism

$$\mathcal{J} : V \oplus V^* \to V \oplus V^*$$

which satisfies $\mathcal{J}^2 = -1$ and $\mathcal{J}^* = -\mathcal{J}$.

Examples

- Let $J\colon V \to V$ be a complex structure on V, then the endomorphism $\mathcal{J}_J\colon V \oplus V^* \to V \oplus V^*$ defined by

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$$

is a generalized complex structure on V.

- Analogously, if we consider a symplectic structure ω on V, then the endomorphism

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

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There is a characterization of generalized complex structures from maximal isotropic subspaces.

Definition

A subspace $L \subset V \oplus V^*$ is said to be isotropic when $\langle X, Y \rangle = 0$ for all $X, Y \in L$. That is, when $L \subset L^\perp$ with respect to the bilinear form $\langle \cdot, \cdot \rangle$. When $\dim L = \dim V = n$ we say that L is a maximal isotropic subspace.

Proposition (Hitchin '03)

A generalized complex structure on V is equivalent to a maximal isotropic subspace $L \subset (V \oplus V^*) \otimes \mathbb{C}$ such that $L \cap \overline{L} = \{0\}$.

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Everything we have done so far can be extended to a smooth manifold M^n.

Definition

A generalized almost complex structure on a manifold M is given by one of the following equivalent data:

- *an almost complex structure J on $TM \oplus T^*M$ which is orthogonal with respect to the natural inner product $\langle \cdot, \cdot \rangle$.*
- *a maximal isotropic subbundle $L < (TM \oplus T^*M) \otimes \mathbb{C}$ of real index zero, that is, $L \cap \overline{L} = \{0\}$.*
- *a pure spinor line subbundle $U < \wedge^\bullet T^*M \otimes \mathbb{C}$ satisfying $(\varphi, \overline{\varphi}) \neq 0$ at each point $x \in M$ for any generator $\varphi \in U_x$.*

A natural question when we are working with these structures is: What about integrability? To talk about integrability we need to define the Courant bracket:

Definition

The Courant bracket is a skew-symmetric bracket defined on the smooth sections of $TM \oplus T^*M$, given by

$$[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi),$$

where $X+\xi, Y+\eta \in \Gamma(TM \oplus T^*M)$.

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Definition

A generalized almost complex structure $\mathcal{J}$ on M is said to be integrable when its i-eigenbundle $L \subset (TM \oplus T^*M) \otimes \mathbb{C}$ is Courant involutive.

Example

- $\mathcal{J}_J$ *is integrable if and only if* $N_J \equiv 0$.
- $\mathcal{J}_\omega$ *is integrable if and only if* $d\omega = 0$.

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- $\mathcal{J}_J$ *is integrable if and only if* $N_J \equiv 0$.
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We also can define the Nijenhuis operator

$$\mathrm{Nij}(A, B, C) = \frac{1}{3}(\langle [A, B], C\rangle + \langle [B, C], A\rangle + \langle [C, A], B\rangle).$$

with $A, B, C \in \Gamma(TM \oplus T^*M)$.

Proposition (Gualtieri '04)

Let $\mathcal{J}$ be a generalized almost complex structure on M. Then $\mathcal{J}$ is integrable if and only if the Nijenhuis operator is zero when restricted to the i-eigenbundle of $\mathcal{J}$.

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Flag Manifolds

- G semi-simple complex Lie group.
- $\mathfrak{g}$ Lie algebra of G.
- $\mathfrak{h}$ Cartan subalgebra of $\mathfrak{g}$.

Thus we have the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Pi} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{h}, [H, X] = \alpha(H)X\}$ is the root space associated to the root α.

- Given $\alpha \in \mathfrak{h}^*$ we define H_α such that $\alpha(\cdot) = \langle H_\alpha, \cdot \rangle$.
- $\mathfrak{h}_\mathbb{R} = span_\mathbb{R}\{H_\alpha \mid \alpha \in \Pi\}$.

We fix a Weyl basis of $\mathfrak{g}$ which is given by elements $X_\alpha \in \mathfrak{g}_\alpha$ such that $\langle X_\alpha, X_{-\alpha} \rangle = 1$, and $[X_\alpha, X_\beta] = m_{\alpha,\beta} X_{\alpha+\beta}$ with $m_{\alpha,\beta} \in \mathbb{R}$, $m_{-\alpha,-\beta} = -m_{\alpha,\beta}$ and $m_{\alpha,\beta} = 0$ if $\alpha + \beta$ is not a root.

- Let $\Pi^+ \subset \Pi$ be a choice of positive roots.
- Denote by Σ the corresponding simple system of roots.

Definition

Define by

$$\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha$$

the Borel subalgebra generated by Π^+. And we say that a subalgebra $\mathfrak{p}$ is a parabolic subalgebra when $\mathfrak{b} \subset \mathfrak{p}$.

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Definition

A flag manifold is the homogeneous space $\mathbb{F} = G/P$ where G is a complex semi-simple Lie group with Lie algebra $\mathfrak{g}$ and P is the normalizer of $\mathfrak{p}$ in G.

If $\mathfrak{p} = \mathfrak{b}$ we have a maximal flag manifold. In this case, we have that

$$T_{b_0}\mathbb{F} = \sum_{\alpha \in \Pi^+} \mathfrak{g}_{-\alpha} = \mathfrak{n}^-.$$

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$$T_{b_0}\mathbb{F} = \sum_{\alpha \in \Pi^+} \mathfrak{g}_{-\alpha} = \mathfrak{n}^-.$$

Let $\mathfrak{u}$ be the compact real form of $\mathfrak{g}$, that is, the real sub-algebra

$$\mathfrak{u} = \mathrm{span}_{\mathbb{R}}\{i\mathfrak{h}_{\mathbb{R}}, A_\alpha, S_\alpha \mid \alpha \in \Pi^+\}$$

where $A_\alpha = X_\alpha - X_{-\alpha}$ and $S_\alpha = i(X_\alpha + X_{-\alpha})$. Denote by $U = \exp \mathfrak{u}$ the corresponding compact real form of G. Then we can write $\mathbb{F} = U/T$ where $T = P \cap U$ is a maximal torus of U.

In this case, we have

$$T_{b_0}\mathbb{F} = \sum_{\alpha \in \Pi^+} \mathfrak{u}_\alpha$$

where $\mathfrak{u}_\alpha = span_{\mathbb{R}}\{A_\alpha, S_\alpha\}$.

Complex structures: (Borel–Hirzebruch '58) Let J be an invariant almost complex structure on $\mathbb{F}$.

- $JX_\alpha = i\varepsilon_\alpha X_\alpha$, where $\varepsilon_\alpha = \pm 1 \leftrightsquigarrow J = \{\varepsilon_\alpha\}$.
- The integrability of J depends on triple of roots satisfying $\alpha + \beta + \gamma = 0$.
- If J is integrable, then the set

$$P = \{\alpha \in \Pi \mid \varepsilon_\alpha = 1\}$$

  is a choice of positive roots with respect to some lexicographic order in $\mathfrak{h}_{\mathbb{R}}^*$.

Invariant Generalized Complex Structures on Flag Manifolds

Consider $M = \mathbb{F}$ a maximal flag manifold.

Problem: Describe the invariant generalized complex structures on $\mathbb{F}$.

- Invariant $\rightsquigarrow$ isotropy representation.
- Since $\mathbb{F}$ is a reductive homogeneous space, then the isotropy representation can be identified with the adjoint representation.
- An invariant generalized complex structure is completely described by its value in the origin

$$\mathcal{J} : \mathfrak{n}^- \oplus (\mathfrak{n}^-)^* \to \mathfrak{n}^- \oplus (\mathfrak{n}^-)^*.$$

Any invariant subspace $L < \mathfrak{n}^- \oplus (\mathfrak{n}^-)^*$ can be decomposed as

$$L = \sum_{\alpha} (\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}^*_{-\alpha}) \cap L.$$

Proposition

Let L be an invariant subspace. Then L is isotropic if and only if $L_\alpha = L \cap (\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}^*_{-\alpha})$ is an isotropic subspace for each α.

To make computations easier it is better to work with the compact real form $\mathfrak{u}$ of $\mathfrak{g}$. So, we have that an invariant maximal isotropic subspace is described by $L = \sum_\alpha L_\alpha$ where $L_\alpha = L \cap (\mathfrak{u}_\alpha \oplus \mathfrak{u}_\alpha^*)$. Which means that an invariant generalized almost complex structure is decomposed as

$$\mathcal{J} = \begin{pmatrix} \mathcal{J}_{\alpha_1} & 0 & \cdots & 0 \\ 0 & \mathcal{J}_{\alpha_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{J}_{\alpha_n} \end{pmatrix}$$

where $\mathcal{J}_{\alpha_k}$ is a generalized complex structure on $\mathfrak{u}_{\alpha_k} \oplus \mathfrak{u}_{\alpha_k}^*$ for each $k \in \{1, \cdots, n\}$.

We want to describe the structures $\mathcal{J}_\alpha$ such that:

- $\mathcal{J}_\alpha^2 = -1$;
- $\langle \mathcal{J}_\alpha x, \mathcal{J}_\alpha y \rangle = \langle x, y \rangle$;
- $\mathcal{J}_\alpha$ is invariant.

After all we concluded that there are only two types of structures satisfying the requirement stated above :

$$\mathcal{J}_\alpha = \pm \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ or } \mathcal{J}_\alpha = \begin{pmatrix} a_\alpha & 0 & 0 & -x_\alpha \\ 0 & a_\alpha & x_\alpha & 0 \\ 0 & -y_\alpha & -a_\alpha & 0 \\ y_\alpha & 0 & 0 & -a_\alpha \end{pmatrix}$$

where $a_\alpha^2 = x_\alpha y_\alpha - 1$.

Therefore, given $\mathcal{J}$ an invariant generalized almost complex structure on a flag manifold $\mathbb{F}$ we have that $\mathcal{J}$ restricted to $\mathfrak{u}_\alpha \oplus \mathfrak{u}_\alpha^*$ is

$$\mathcal{J}_\alpha = \pm J_0 = \pm \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{complex type}$$

or

$$\mathcal{J}_\alpha = \begin{pmatrix} a_\alpha & 0 & 0 & -x_\alpha \\ 0 & a_\alpha & x_\alpha & 0 \\ 0 & -y_\alpha & -a_\alpha & 0 \\ y_\alpha & 0 & 0 & -a_\alpha \end{pmatrix} \quad \text{noncomplex type}$$

where $a_\alpha^2 = x_\alpha y_\alpha - 1$.

With simple calculations we can prove that Nij is zero, except possibly when we have a triple of roots $(\alpha, \beta, \alpha + \beta)$.

$\mathcal{J}_\alpha$	$\mathcal{J}_\beta$	$\mathcal{J}_{\alpha+\beta}$
complex $(\pm J_0)$	complex $(\pm J_0)$	complex $(\pm J_0)$
complex $(\pm J_0)$	complex $(\mp J_0)$	complex $(\pm J_0)$
complex $(\pm J_0)$	complex $(\mp J_0)$	complex $(\mp J_0)$
noncomplex	complex $(\pm J_0)$	complex $(\pm J_0)$
complex $(\pm J_0)$	noncomplex	complex $(\pm J_0)$
complex $(\pm J_0)$	complex $(\mp J_0)$	noncomplex
noncomplex	noncomplex	noncomplex

where, for the last row, we have the additional conditions

$$\begin{cases} a_{\alpha+\beta} x_\alpha x_\beta - a_\beta x_\alpha x_{\alpha+\beta} - a_\alpha x_\beta x_{\alpha+\beta} = 0 \\ x_\alpha x_\beta - x_\alpha x_{\alpha+\beta} - x_\beta x_{\alpha+\beta} = 0 \end{cases}$$

Lemma (V.–San Martin '20)

Let $\mathcal{J}$ be an integrable invariant generalized complex structure on $\mathbb{F}$. Then the set

$$P = \{\alpha \in \Pi \mid \mathcal{J}_\alpha \text{ is of complex type with } \mathcal{J}_\alpha = J_0\} \cup \{\alpha \in \Pi \mid \mathcal{J}_\alpha \text{ is of noncomplex type with } x_\alpha > 0\}$$

is a choice of positive roots with respect to some lexicographic order in $\mathfrak{h}^*_{\mathbb{R}}$.

Theorem (V.–San Martin '20)

Let $\mathcal{J}$ be an integrable invariant generalized complex structure on $\mathbb{F}$. Then there is a simple root system Σ and a subset $\Theta \subset \Sigma$, such that $\mathcal{J}_\alpha$ is of noncomplex type for $\alpha \in \langle \Theta \rangle$ and of complex type for $\alpha \in \Pi \backslash \langle \Theta \rangle$.

Reciprocally:

Theorem (V.–San Martin '20)

Let Σ be a simple root system and let $\Theta \subset \Sigma$ be a subset. Then there is an integrable invariant generalized complex structure $\mathcal{J}$ on $\mathbb{F}$ such that $\mathcal{J}_\alpha$ is of noncomplex type for $\alpha \in \langle \Theta \rangle$ and of complex type for $\alpha \in \Pi \backslash \langle \Theta \rangle$.

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• VAREA, C. A. B.. Invariant generalized complex structures on partial flag manifolds. **Indagationes Mathematicae**, v. 31, p. 536-555, 2020.

Thank you!