The Poisson saturation of regular submanifolds

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Motivation

Symplectic geometry

If $X \subset (M, \omega)$ is any submanifold, then $\omega|_X$ determines ω around X up to symplectomorphism.

Poisson geometry

If $X \subset (M, \pi)$, then $\pi|_X$ does not determine π around X up to Poisson diffeomorphism, e.g.:

$$(x^2 + y^2)\partial_x \wedge \partial_y \not\simeq 0.$$

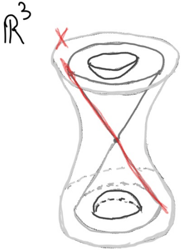
$\pi|_X$ only sees leaves intersecting X...

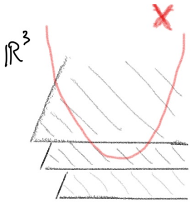
The saturation

Definition

The saturation of $X \subset (M,\pi)$ is the union of leaves intersecting X.

⚠ $Sat(X)$ is not a submanifold for arbitrary $X \subset (M,\pi)$...





When smooth, $Sat(X)$ is the smallest complete Poisson submanifold containing X.

Regular submanifolds

Definition

$X \subset (M,\pi)$ is regular if $pr \circ \pi^\sharp : T^*M|_X \to NX$ has constant rank.

Examples

- Any submanifold of a symplectic manifold
- Transversals
- Poisson submanifolds

For any $X \subset (M,\pi)$, denote $TX^{\perp_\pi} := \pi^\sharp(TX^0)$. If (S,ω) is the leaf through $x \in X$, then

$$T_xX^{\perp_\pi} = (T_xX \cap T_xS)^{\perp_\omega}.$$

Observation

$X \subset (M,\pi)$ is regular $\Leftrightarrow$ $TX^{\perp_\pi}$ has constant rank.

Regular submanifolds

Remarks

- If $X \subset (M, \pi)$ is regular, then the pullback of L_π is a smooth *Dirac* structure.

  (Think of a Dirac structure as a singular foliation with a closed two-form on each leaf.)

- There is a notion of *coregular*[1] submanifold $X \subset (M, \pi)$, defined by requiring that

$$TX \oplus TX^{\perp_\pi} \subset TM|_X.$$

  The pullback of L_π is a smooth *Poisson* structure.

[1] L. Brambila, P. Frejlich and D. Martinez-Torres, *Coregular submanifolds and Poisson submersions*, preprint ArXiv:2010.09058 (2020).

The saturation of a regular submanifold

Theorem A

A regular submanifold $X \subset (M, \pi)$ has a neighborhood U such that

$$Sat_{loc}(X) := Sat(X \text{ in } (U, \pi|_U))$$

is an embedded Poisson submanifold of (M, π).

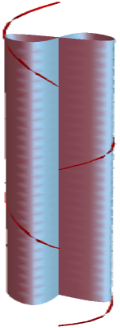
In general, there is no natural smooth structure on $Sat(X)$.

Example

Take $(\mathbb{R}^3 \times S^1, \partial_z \wedge \partial_\theta)$ with $X := \mathscr{C} \times S^1$.
Here $\mathscr{C} \subset \mathbb{R}^3$ is the image of

$$\beta : \mathbb{R} \to \mathbb{R}^3 : t \mapsto (\sin(2t), \sin(t), t).$$

Then X is regular and $Sat(X) = \infty \times \mathbb{R} \times S^1$.
$Sat(X)$ has two natural smooth structures, but
$X \hookrightarrow Sat(X)$ is not continuous for either...



The local model for $(Sat_{loc}(X),\pi)$

Lives on $(TX^{\perp_\pi})^*$, and involves 2 choices (W,η):

- A complement $TM|_X = TX^{\perp_\pi} \oplus W$ gives an inclusion
$$j:(TX^{\perp_\pi})^* \hookrightarrow T^*M|_X.$$
- A closed two-form η on a nbhd of $X \subset (TX^{\perp_\pi})^*$ satisfying
$$\eta|_X\big((v_1,\xi_1),(v_2,\xi_2)\big) = -\pi\big(j(\xi_1),j(\xi_2)\big) - \langle v_1, j(\xi_2)\rangle + \langle v_2, j(\xi_1)\rangle.$$
Here $(v_1,\xi_1),(v_2,\xi_2) \in T(TX^{\perp_\pi})^*|_X = TX \oplus (TX^{\perp_\pi})^*$.

The model is
$$\Big((TX^{\perp_\pi})^*, (pr^*(i^*L_\pi))^\eta\Big).$$

Poisson near X, and independent of choices up to isomorphism.

The normal form for $(Sat_{loc}(X),\pi)$

Theorem B

A neighborhood of X in $(Sat_{loc}(X),\pi)$ is Poisson diffeomorphic with the local model

$$\Big((TX^{\perp_\pi})^*,(pr^*(i^*L_\pi))^\eta\Big).$$

Remark

Note that $X\subset(Sat_{loc}(X),\pi)$ is a transversal. Our Theorem B agrees with the normal form around Dirac transversals[2].

The local model only depends on $\pi|_X$!

Corollary

If $X\subset(M,\pi)$ is regular, then $\pi|_X$ determines $(Sat_{loc}(X),\pi)$ around X up to Poisson diffeomorphism.

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[2]H. Bursztyn, H. Lima and E. Meinrenken, *Splitting theorems for Poisson and related structures*, J. Reine Angew. Math. **754** (2019), p. 281-312.

Special case: regular coisotropic submanifolds

Recall: Gotay normal form

Let $X \subset (M,\omega)$ be coisotropic . A complement $TX = TX^{\perp_\omega} \oplus G$ gives an inclusion $j : (TX^{\perp_\omega})^* \hookrightarrow T^*X$. Then

$$(M,\omega) \cong \left((TX^{\perp_\omega})^*, pr^*(i^*\omega) + j^*\omega_{can}\right).$$

Let $X \subset (M,\pi)$ be regular and coisotropic, i.e. $TX^{\perp_\pi} \subset TX$. Good choice for data (W,η) makes normal form more explicit.

Poisson Gotay

Let $X \subset (M,\pi)$ be regular coisotropic, and fix $TX = TX^{\perp_\pi} \oplus G$. Denoting $j : (TX^{\perp_\pi})^* \hookrightarrow T^*X$, we get

$$\left(Sat_{loc}(X), \pi\right) \cong \left((TX^{\perp_\pi})^*, (pr^*(i^*L_\pi))^{j^*\omega_{can}}\right).$$

Application

Recall: coisotropic embeddings in symplectic manifolds

(X,ω) manifold with closed two-form.

- $(X,\omega) \hookrightarrow (M,\Omega)$ exists iff. ω has constant rank.
- (M,Ω) is unique up to neighborhood equivalence.

Coisotropic embeddings of Dirac (X,L) into Poisson (M,π)?[3]

Existence ✓

$(X,L) \hookrightarrow (M,\pi)$ exists iff. $L \cap TX$ has constant rank.

A complement $TX = (L \cap TX) \oplus G$ gives $j : (L \cap TX)^* \hookrightarrow T^*X$. Set

$$(M,\pi) := \left((L \cap TX)^*, (pr^* L)^{j^* \omega_{can}}\right).$$

[3] A. Cattaneo and M. Zambon, *Coisotropic embeddings in Poisson manifolds*, Trans. Amer. Math. Soc. **361** (2009), p. 3721-3746.

Uniqueness ?

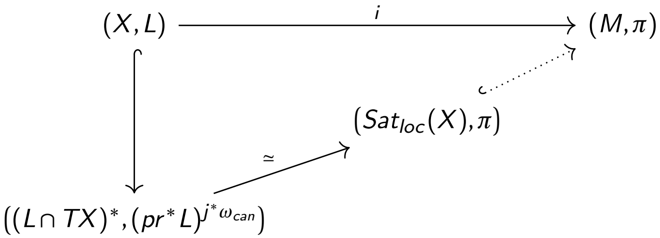
Conjecture: (M,π) is unique up to neighborhood equivalence, provided that $\dim M = \dim X + rk(L \cap TX)$. [Cattaneo-Zambon]

If $i:(X,L) \hookrightarrow (M,\pi)$, then X is regular:

$$TX^{\perp_\pi} = \pi^\sharp(TX^0) = (i^* L_\pi) \cap TX = L \cap TX.$$

Corollary

- If $i:(X,L) \hookrightarrow (M,\pi)$, then



- Uniqueness if $\dim M = \dim X + rk(L \cap TX)$.