Symplectic excision

Xiudi Tang
University of Toronto
arXiv:2101.03534 (joint with Yael Karshon)

Junior Global Poisson Workshop
May 5, 2021

Section 1

Symplectic excisibility conditions

Excisability conditions

Question

Let (M, ω) be a symplectic manifold with a closed subset Z.

- Is there a diffeomorphism $\varphi\colon M \setminus Z \to M$?
- Is there a symplectomorphism $\varphi\colon M \setminus Z \to M$?

- Is there a symplectomorphism supported in an arbitrary neighborhood of Z?
- Can this symplectomorphism be realized as a Hamiltonian flow on M whose forward flow preserves Z?
- Can this Hamiltonian symplectomorphism be time-independent?

Excisability conditions

Question

Let (M, ω) be a symplectic manifold with a closed subset Z.

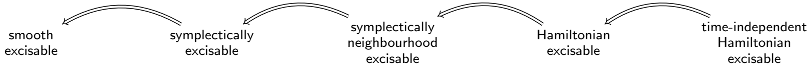
- Is there a diffeomorphism $\varphi \colon M \setminus Z \to M$? (smoothly excisable)
- Is there a symplectomorphism $\varphi \colon M \setminus Z \to M$? (symplectically excisable) $\to \varphi$ is a symplectic excision of Z from M
- Is there a symplectomorphism supported in an arbitrary neighborhood of Z? (symplectically neighbourhood excisable)
- Can this symplectomorphism be realized as a Hamiltonian flow on M whose forward flow preserves Z? (Hamiltonian excisable)
- Can this Hamiltonian symplectomorphism be time-independent? (time-independently Hamiltonian excisable)

Excisability conditions

Question

Let (M, ω) be a symplectic manifold with a closed subset Z.

- Is there a diffeomorphism $\varphi\colon M \setminus Z \to M$? (smoothly excisable)
- Is there a symplectomorphism $\varphi\colon M \setminus Z \to M$? (symplectically excisable) $\to \varphi$ is a symplectic excision of Z from M
- Is there a symplectomorphism supported in an arbitrary neighborhood of Z? (symplectically neighbourhood excisable)
- Can this symplectomorphism be realized as a Hamiltonian flow on M whose forward flow preserves Z? (Hamiltonian excisable)
- Can this Hamiltonian symplectomorphism be time-independent? (time-independently Hamiltonian excisable)



Having a smooth excision is not sufficient.

(Eliashberg–Gromov 1990, McDuff–Traynor 1993) considered the symplectic camel space which is the complement of the wall with a hole

$$Z_s = \left\{ z \;\middle|\; x_1 = 0, \sum_{i=1}^{n} \left(x_i^2 + y_i^2\right) \geqslant s \right\} \subset (\mathbb{R}^{2n}, \omega_{\mathrm{can}}).$$

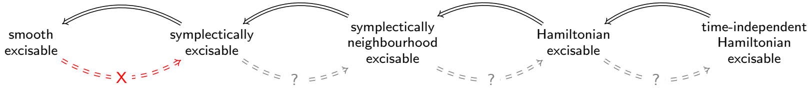
There is no symplectic excision of Z_s for $s > 0$ from $(\mathbb{R}^{2n}, \omega_{\mathrm{can}})$ since no ball of radius $> s$ can go through the hole in the wall.

Having a smooth excision is not sufficient.

(Eliashberg–Gromov 1990, McDuff–Traynor 1993) considered the symplectic camel space which is the complement of the wall with a hole

$$Z_s = \left\{ z \;\middle|\; x_1 = 0, \sum_{i=1}^{n} \left(x_i^2 + y_i^2\right) \geqslant s \right\} \subset (\mathbb{R}^{2n}, \omega_{\mathrm{can}}).$$

There is no symplectic excision of Z_s for $s > 0$ from $(\mathbb{R}^{2n}, \omega_{\mathrm{can}})$ since no ball of radius $> s$ can go through the hole in the wall.

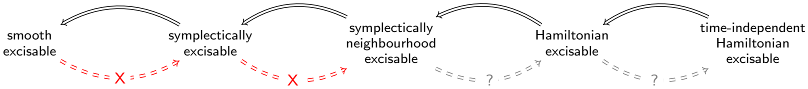


Zero volume condition.

If Z is symplectically neighbourhood excisable from (M, ω) then Z has zero ω-symplectic volume. Therefore, the half space $Z = \left\{ z \in \mathbb{R}^{2n} \,\middle|\, x_n \geqslant 0 \right\}$ is symplectically excisable from $(\mathbb{R}^{2n}, \omega_{\mathrm{can}})$ but is not symplectically neighbourhood excisable from it.

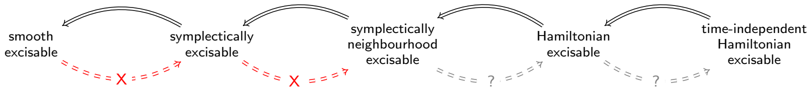
Zero volume condition.

If Z is symplectically neighbourhood excisable from (M, ω) then Z has zero ω-symplectic volume. Therefore, the half space $Z = \left\{ z \in \mathbb{R}^{2n} \;\middle|\; x_n \geqslant 0 \right\}$ is symplectically excisable from $(\mathbb{R}^{2n}, \omega_{\mathrm{can}})$ but is not symplectically neighbourhood excisable from it.



Zero volume condition.

If Z is symplectically neighbourhood excisable from (M, ω) then Z has zero ω-symplectic volume. Therefore, the half space $Z = \left\{ z \in \mathbb{R}^{2n} \mid x_n \geqslant 0 \right\}$ is symplectically excisable from $(\mathbb{R}^{2n}, \omega_{\mathrm{can}})$ but is not symplectically neighbourhood excisable from it.



What about the last three excisibility conditions?
Are they equivalent to each other?

Section 2

Examples

Example 1

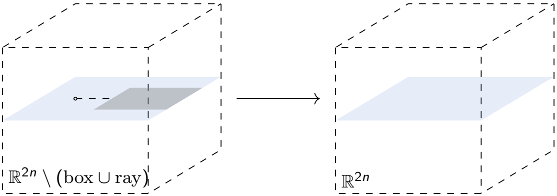
Example (Ray, Karshon–T. [2])

A properly embedded ray $[0, +\infty)$ is time-independently Hamiltonian excisable from a (noncompact) symplectic manifold.

Example 1

Example (Ray, Karshon–T. [2])

A properly embedded ray $[0, +\infty)$ is time-independently Hamiltonian excisable from a (noncompact) symplectic manifold.



Example 2

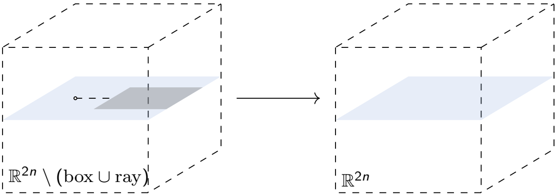
Example (Box with a tail)

The box with a tail
$Z = [-1,1]^{2n-2} \times [0,\infty) \times \{0\} \cup \{0\}^{2n-2} \times [-1,\infty) \times \{0\}$ is time-independently Hamiltonian excisable from $(\mathbb{R}^{2n}, \omega_{\mathrm{can}})$ when $k \geqslant 2$.

Example 2

Example (Box with a tail)

The box with a tail
$Z = [-1,1]^{2n-2} \times [0,\infty) \times \{0\} \cup \{0\}^{2n-2} \times [-1,\infty) \times \{0\}$ is time-independently Hamiltonian excisable from $(\mathbb{R}^{2n}, \omega_{\text{can}})$ when $k \geqslant 2$.



Example 3

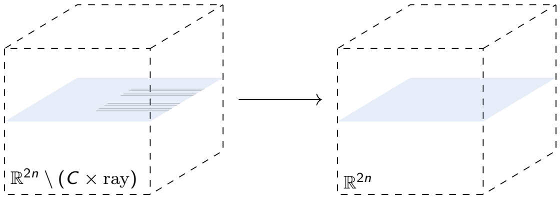
Example (Cantor brush)

The Cantor brush $\{0\}^{2n-2} \times C \times [0, \infty)$, where C is the Cantor set, is time-independently Hamiltonian excisable from $(\mathbb{R}^{2n}, \omega_{\mathrm{can}})$.

Example 3

Example (Cantor brush)

The Cantor brush $\{0\}^{2n-2} \times C \times [0, \infty)$, where C is the Cantor set, is time-independently Hamiltonian excisable from $(\mathbb{R}^{2n}, \omega_{\mathrm{can}})$.



Example 4

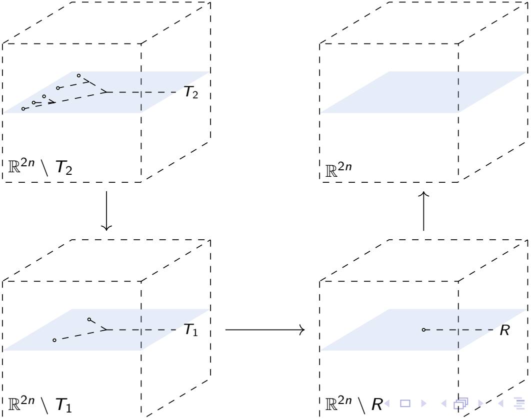
Example (Open-rooted tree)

Any properly embedded finite open-rooted tree T in a symplectic manifold (M, ω) is Hamiltonian excisable from it.

Example 4

Example (Open-rooted tree)

Any properly embedded finite open-rooted tree T in a symplectic manifold (M, ω) is Hamiltonian excisable from it.



Section 3

Removing a ray

Removing a ray

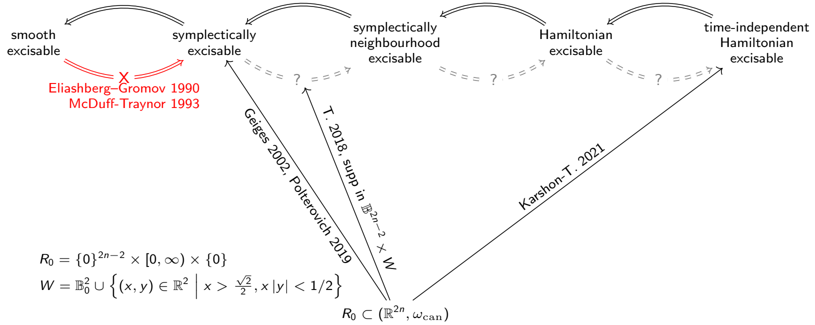
Theorem (Karshon–T. [2])

Any (noncompact) symplectic manifold is symplectomorphic to its complement of any properly embedded ray. Moreover, the symplectomorphism can be given by a time-independent Hamiltonian flow.

Removing a ray

Theorem (Karshon–T. [2])

Any (noncompact) symplectic manifold is symplectomorphic to its complement of any properly embedded ray. Moreover, the symplectomorphism can be given by a time-independent Hamiltonian flow.



Time-independent Hamiltonian flow

Let (M, ω) be a symplectic manifold with a smooth function $F \colon M \to \mathbb{R}$. Then F generates

- a Hamiltonian vector field X_F,
- a Hamiltonian flow $\Phi_F \colon D_F \to I$,
- the flow domain $D_F \subseteq \mathbb{R} \times M$ given by

$$D_F = \{(t, x) \in \mathbb{R} \times M \mid S_F(x) < t < T_F(x)\},$$

- the backward time function $S_F \colon M \to [-\infty, 0)$ (upper semi-continuous),
- the forward time function $T_F \colon M \to (0, \infty]$ (lower semi-continuous),
- subsets $Y_t = \{S_F \geqslant -t\}$ and $Z_t = \{T_F \leqslant t\}$ of M,
- and symplectomorphisms $\varphi_t \colon (M \setminus Z_t, \omega|_{M \setminus Z_t}) \to (M \setminus Y_t, \omega|_{M \setminus Y_t})$.

The time-1 map as the excision

Suppose

- the forward flow of X_F preserves Z;
- $T_F > 1$ on $M \setminus Z$;
- $T_F \leqslant 1$ on Z;
- $S_F < -1$ everywhere on M.

Then

- $Y_1 = \emptyset$,
- $Z_1 = Z$,
- and the time-1 map $\varphi_t \colon (M \setminus Z, \omega|_{M\setminus Z}) \to (M, \omega)$ is a time-independently Hamiltonian excision of Z from (M, ω).

Removing a ray from $(\mathbb{R}^{2n}, \omega_{\mathrm{can}})$

Theorem

R_0 is time-independently Hamiltonian excisable from $(\mathbb{R}^{2n}, \omega_{\mathrm{can}})$.

Removing a ray from $(\mathbb{R}^{2n}, \omega_{\mathrm{can}})$

Theorem

R_0 is time-independently Hamiltonian excisable from $(\mathbb{R}^{2n}, \omega_{\mathrm{can}})$.

Sketch of proof.

Equivalently, we excise

$$R_1 = \{0\}^{2n-2} \times [0,1) \times \{0\} \quad \text{from} \quad M := \mathbb{R}^{2n-2} \times (-1,1) \times \mathbb{R},$$

with coordinates $z = (p; x_n, y_n)$ where $p = (x_1, y_1, \ldots, x_{n-1}, y_{n-1})$. Fix a bump function $\chi \colon M \to [0,1]$ supported in a suitable neighbourhood of R_1 and equal to 1 in a larger neighbourhood of R_1 in M. Let

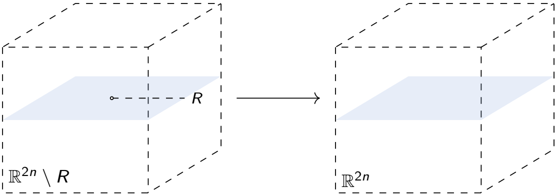
$$F(z) := \frac{1 - x_n^2}{|p|^2 + 1 - x_n^2} \chi(z) y_n.$$

□

Removing a ray from $(\mathbb{R}^{2n}, \omega_{\text{can}})$

Theorem

R_0 is time-independently Hamiltonian excisable from $(\mathbb{R}^{2n}, \omega_{\text{can}})$.



Section 4

The excision theorem

Theorem (Karshon–T. [2])

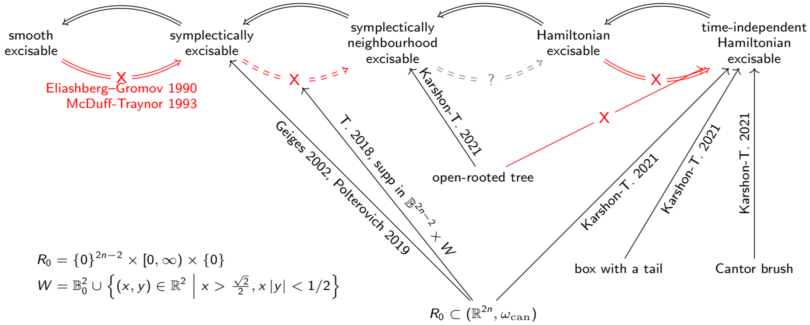
Let (M, ω) be a $2n$-dimensional symplectic manifold. Let (B, ω_B) be a $(2n-2)$-dimensional symplectic manifold. Let $\lambda\colon B \to (0,1]$ be a lower semi-continuous function. Let $U_0 \subseteq B \times I$ be an open neighbourhood of the epigraph $Z_0 = \{(p,x) \in B \times I \mid x \geqslant \lambda(p)\}$. Let $\psi\colon U_0 \to M$ be an embedding such that $\psi^*\omega = \omega_B \oplus 0$ and $Z = \psi(Z_0)$ is closed in M. Then Z is time-independently Hamiltonian excisable from (M, ω).

Sketch of proof.

Construct a null vector field X on $B \times I$ with $Y_1 = \emptyset$ and Z_1 be the epigraph. Then extend the null vector field X to a Hamiltonian vector field X_F on M. □

Section 5

Further applications



$R_0 = \{0\}^{2n-2} \times [0, \infty) \times \{0\}$

$W = \mathbb{B}^2_0 \cup \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; x > \frac{\sqrt{2}}{2}, x\,|y| < 1/2 \right\}$

Application 1

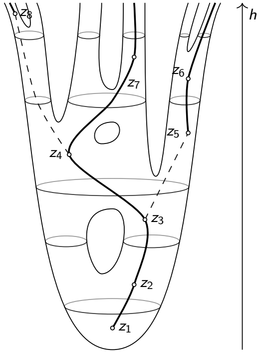
Corollary (Non-vanishing Liouville form)

Any exact symplectic form has a nowhere vanishing primitive. That is, if $(M, \omega = \mathrm{d}\theta)$ *is an exact symplectic manifold then there is a nowhere vanishing* 1*-form* α *on* M *such that* $\omega = \mathrm{d}\alpha$*.*

Application 1

Corollary (Non-vanishing Liouville form)

Any exact symplectic form has a nowhere vanishing primitive. That is, if $(M, \omega = \mathrm{d}\theta)$ is an exact symplectic manifold then there is a nowhere vanishing 1*-form* α *on* M *such that* $\omega = \mathrm{d}\alpha$.



Application 2

Example (Retracting a tree)

Let (M, ω) be a symplectic manifold with a compact connected embedded unrooted tree T. Then for any open neighbourhood U of T in M there is a symplectomorphism $\varphi\colon (M \setminus T, \omega) \to (M \setminus \{z_0\}, \omega)$ supported in U for any $z_0 \in T$.

Application 2

Example (Retracting a tree)

Let (M, ω) be a symplectic manifold with a compact connected embedded unrooted tree T. Then for any open neighbourhood U of T in M there is a symplectomorphism $\varphi\colon (M \setminus T, \omega) \to (M \setminus \{z_0\}, \omega)$ supported in U for any $z_0 \in T$.