"ETH" IN QUANTUM MECHANICS

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1. Introduction

Integrable systems, Bohr – Sommerfeld
Rydberg – Ritz, Thomas – Kuhn

⇓ Heisenberg, Born, Jordan 1925

Matrix Mechanics

⇓ + Dirac, Schrödinger, Pauli

NR Quantum Mechanics

Jordan – Wigner → Atomism

→ Diff. geometry
Spin geometry
SUSY

→ Most of modern
low-energy physics
& high technology

→ A deep puzzle as to
the meaning of QM

Planck's Law

Spect. energy density, according to Planck, 1900:

$$\boxed{\rho(\nu, T) = \frac{8\pi}{c^3} \nu^2 \frac{h\nu}{e^{h\nu/kT} - 1}} \qquad (*)$$

Limiting laws

(i) $h\nu \ll kT$

$$\Rightarrow \rho(\nu, T) \simeq \frac{8\pi\nu^2}{c^3} kT$$

Rayleigh-Jeans, 1900

(ii) $h\nu \gg kT$

$$\Rightarrow \rho(\nu, T) \simeq \frac{8\pi h\nu^3}{c^3} e^{-h\nu/kT}$$

Wien 1896

(New) constants of Nature appearing in (*):

c: speed of light

k: Boltzmann's constant

h: Planck's constant

$k = 1.381 \times 10^{-23}\ J/K$

$h = 6.625 \times 10^{-34}\ J\,sec$

$$\begin{cases} N = \dfrac{R}{k} = 6.022 \times 10^{23}\ /Mol \\ \\ e = \dfrac{F}{N} = 1.602 \times 10^{-19}\ \text{Coulombs} \\ \quad = 4.803 \times 10^{-10}\ \text{statcoulombs (cgs)} \end{cases}$$

$$l_p^2 := \frac{G_N \hbar}{c^3}$$

$l_p \simeq 1.6 \times 10^{-33}\ cm$; Planck length.

Revolutions in 20th Century Physics

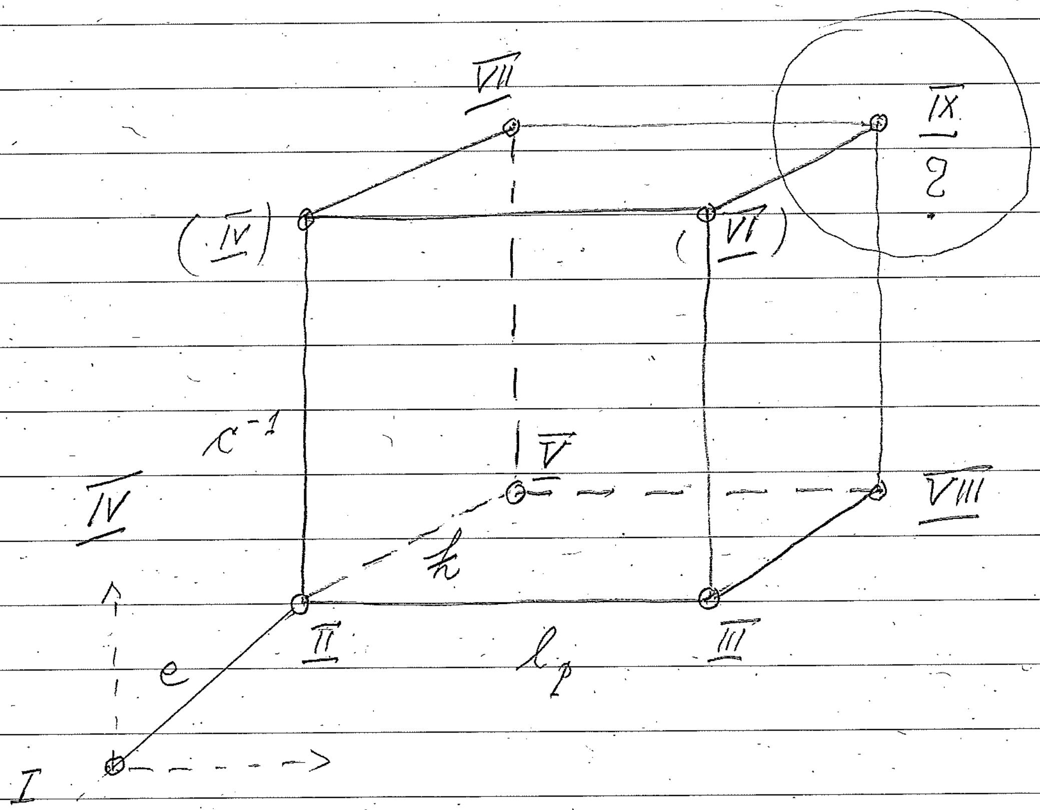
$c \longleftrightarrow$ special relativity

$e \longleftrightarrow$ atomism

$h \longleftrightarrow$ quantum theory

$l_p \longleftrightarrow$ Space, time and gravitation

Bronstein–Planck (Hyper) Cube



I : Mechanics of Continuous Media, Thermodynamics

II : Hamiltonian mech. and statistical mech. of point particles

III : Celestial mechanics; Newtonian space, time and gravitation

IV : Classical (special) relativistic field theory, electrodynamics

$\boxed{\text{V : NR quantum mechanics of point particles}}$

VI : General Relativity

$\boxed{\text{VII : Relativistic QFT}}$

VIII : NR Quantum Gravity

IX : "String- and M-theory"

c^{-1}, e, h, l_P : "deformation parameters"

Examples: Galilei $\xrightarrow{c^{-1}}$ Poincaré Ex. 1

Hamiltonian mech. $\xrightarrow{h}$ matrix mech. Ex. 2

continuum mech. $\xrightarrow{e}$ atomism Ex. 3

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2. What is the fundamental problem in QM?

2.1 Conventional formulation of QM.

Conventionally, theory formulated in terms of

$\mathcal{S}$

- Hilbert space of pure state vectors
- Unitary propagator describing time evolution
- "Pictures": Configuration-space rep.
  momentum-space rep.
  energy rep.

+ transformations between different pictures (Dirac)

Missing: Dictionary between structure $\mathcal{S}$ and processes in Nature.

- In particular, we don't know what an "event" is and how it is described in quantum mechanics.
- Where does fundamentally probabilistic nature of QM come from? No mention of probabilities in $\mathcal{S}$; so how do they enter the formalism?
- What is the meaning of "probabilities" in a theory of an evolving world where situations do not repeat?

↓

Copenhagen mumbo jumbo

2.2. A formulation of classical physics reminiscent of structure $\mathcal{S}$.

See transparences!

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3. A simple analysis of the impossibility of hidden variables.

Einstein, Schrödinger, von Laue, ...;

Final Theory should give a realistic description of Nature; probabilities should enter as in classical theories; as an expression of ignorance.

Does not mean that theories are predictive.

Example: Class. Relat. Field Theories — Impossibility to determine initial condition!

3.1 Kochen-Specker

Theories with hidden variables.

Accept predictions of QM according to

Copenhagen interpretation;

Hilbert space $\mathcal{H}$, Obs. represented by

s.a. linear operators, $A = A^*$, acting on $\mathcal{H}$.

Given state $\Psi \in \mathcal{H}$, observable $A = A^*$ (bounded),

$$\langle \Psi, E^A(\Delta)\Psi \rangle =: \mathrm{Prob}_\Psi\{A \in \Delta\}$$

is probability to find a value in

set $\Delta \subset \mathbb{R}$ when A is measured.

If A has value in Δ when A is measured

state right after measurement of A is

$$\frac{E^A(\Delta)\Psi}{\|E^A(\Delta)\Psi\|},$$

("collapse postulate").

Hidden-variables theory:

$\exists$ measure space $(\Omega, \mathcal{F})$ and

maps:

$$A = A^* \text{ Obs.} \longmapsto f_A : \Omega \to \mathbb{R}$$

$$\Psi \in \mathcal{H} \longmapsto \rho_{[\Psi]} \text{ ; prob. measure on } (\Omega, \mathcal{F})$$

with properties:

(P1) $$\boxed{\langle \Psi, E^A(\Delta)\Psi \rangle = \rho_{[\Psi]}(f_A^{-1}(\Delta)), \quad \Delta \subset \mathbb{R} \text{ (Borel set)}}$$

$$\Rightarrow \langle \Psi, A\Psi \rangle = \int_\Omega f_A(\omega)\, d\rho_{[\Psi]}(\omega)$$

(P2) If $u : \mathbb{R} \to \mathbb{R}$ is a bounded measurable function then

$$\boxed{f_{u(A)} = u \circ f_A}$$

(P1) & (P2) are compatible:

$$\langle \Psi, E^{u(A)}(\Delta)\Psi \rangle = \langle \Psi, E^{A}(u^{-1}(\Delta))\Psi \rangle$$

$$\overset{(P1)}{=} \rho_{[\Psi]}\left(f_A^{-1}\left(u^{-1}(\Delta)\right)\right)$$

$$= \rho_{[\Psi]}\left((u \circ f_A)^{-1}(\Delta)\right)$$

$$\overset{(P2)}{=} \rho_{[\Psi]}\left(f_{u(A)}^{-1}(\Delta)\right)$$

(P3) Given any abelian algebra $\mathfrak{M}$ of commuting, self-adjoint operators

$$f : A \in \mathfrak{M} \longmapsto f_A \in L^\infty(\Omega)$$

is an algebra homomorphism; i.e.,

$$f_{A_1 A_2} = f_{A_1} \cdot f_{A_2},$$

for all A_1, A_2 in $\mathfrak{M}$.

(Claim: (P3) follows from (P1), (P2).
If $\dim \mathcal{H} < \infty$ this is easy to prove: Ex.!)

If $\dim \mathcal{H} = 2$ a hidden-variables theory with (P1) – (P3) exists.

Theorem. (Kochen & Specker)

If $\dim \mathcal{H} \geq 3$ a hidden-variables theory with (P1) – (P3) does not exist.

Proof. (following N. D. Mermin)

3 spin-$\frac{1}{2}$ particles:

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$$

6 observables:

$$A_1 = \sigma_x \otimes 1 \otimes 1, \quad A_2 = 1 \otimes \sigma_x \otimes 1, \quad A_3 = 1 \otimes 1 \otimes \sigma_x$$

$$B_1 = \sigma_y \otimes 1 \otimes 1, \quad B_2 = 1 \otimes \sigma_y \otimes 1, \quad B_3 = 1 \otimes 1 \otimes \sigma_y$$

Consider

$$Q_1 = A_1 B_2 B_3, \quad Q_2 = B_1 A_2 B_3, \quad Q_3 = B_1 B_2 A_3$$

(i) The Q_i's are s.a. ops.

$$\sigma_x \sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$\sigma_y \sigma_x = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\Rightarrow \boxed{\sigma_x \sigma_y = - \sigma_y \sigma_x}$$

$$\Rightarrow \text{(ii)} \quad [Q_i, Q_j] = 0, \quad \forall\, i,j$$

Finally,

$$\text{(iii)} \quad Q_i^2 = 1, \quad \forall i.$$

Construct a joint eigenvector with e.v. = 1:

$$\Psi_1 := \frac{1}{\sqrt{2}} \Big[|\uparrow\rangle \otimes |\uparrow\rangle \otimes |\uparrow\rangle - |\downarrow\rangle \otimes |\downarrow\rangle \otimes |\downarrow\rangle \Big]$$

$$\Rightarrow \text{(iv)} \quad Q_j \Psi_1 = \Psi_1, \quad \forall j.$$

Assume that $\exists$ hidden-variable theory reproducing this system:

Def. $q_j := f_{Q_j}$

Then

$$(v) \quad \int_\Omega q_j(\omega)\, d\rho_{[\Psi_1]}(\omega) \overset{(P1)}{=} \langle \Psi_1, Q_j \Psi_1 \rangle \overset{(iii)}{=} 1$$

Because $Q_j^2 = 1 \Rightarrow q_j^2 = 1 \Rightarrow$

$$q_j(\omega) = \pm 1, \quad \text{a.e.}$$

With (v) $\Rightarrow$ $\quad q_j(\omega) = 1, \quad$ a.e.

$$\Rightarrow \quad q_1 q_2 q_3 = 1$$

Because $A_1, B_2 B_3$ commute (+ cycl.!)

$$\Rightarrow$$

$$(vi) \quad q_1 = f_{A_1} f_{B_2} f_{B_3} = a_1 b_2 b_3, \quad + \text{cycl.}$$

(vii) $q_1 q_2 q_3 = a_1 b_2 b_3 \, b_1 a_2 b_3 \, b_1 b_2 a_3$

$= a_1 a_2 a_3 \,,$

because $b_i^2 = f_{B_i}^2 = f_{B_i^2} = f_{\dots\sigma_y^2\dots} = 1$.

But $Q_1 Q_2 Q_3 = -A_1 A_2 A_3$

$\Longrightarrow$

$$q_1 q_2 q_3 = f_{Q_1 Q_2 Q_3} = f_{-A_1 A_2 A_3}$$

$$= -f_{A_1 A_2 A_3} = -f_{A_1} f_{A_2} f_{A_3}$$

$$= -a_1 a_2 a_3$$

$\longrightarrow$ $\boxed{\text{Contradiction!}}$

The original Kochen-Specker argument

$\mathcal{H} = \mathbb{C}^3$, spin-1 particle.

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The operators

$$1-S_1^2=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad 1-S_2^2=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\quad 1-S_3^2=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$$

are mutually commuting projections of rank 1 whose sum $= 1_3$. More generally,

$$P(\vec{e}) = 1-(\vec{e}\cdot\vec{S})^2, \quad |\vec{e}|=1,$$

is a rank-1 proj. projecting onto $\vec{e}$ $\Rightarrow$

$$P(\vec{e}) = |\vec{e}\rangle\langle\vec{e}|$$

$$\Rightarrow \quad P(\vec{e})_{ij} = e_i e_j$$

If $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is any orthonormal dreibein then

(viii) $$\sum_{j=1}^{3} P(\vec{e}_j) = 1, \quad P(\vec{e}_i)\,P(\vec{e}_j) = \delta_{ij}\,P(\vec{e}_i)$$

The ops. $P(\vec{e}_1), P(\vec{e}_2), P(\vec{e}_3)$ are functions of

$$A = \sum_{j=1}^{3} \alpha_j\, P(\vec{e}_j), \qquad \alpha_1 < \alpha_2 < \alpha_3 .$$

→ Can use (P3)! (See page 12.)

Suppose Kochen – Specker hypotheses (P1) – (P3) are correct. Then

(ix) $P(\vec{e}) \longmapsto f_{P(\vec{e})} = \chi_{\vec{e}}$,

a characteristic function on $(\Omega, \mathcal{F})$.

By (viii), page 17,

(x) $$\sum_{j=1}^{3} \chi_{\vec{e}_j} = 1, \quad \text{a.e.}$$

If $d\mu$ is any probability measure on $(\Omega, \mathcal{F})$ then

$$\mathbb{E}_\mu \chi_{\vec{e}} := \int \chi_{\vec{e}}(\omega)\, d\mu(\omega)$$

satisfies

$$0 \leq \mathbb{E}_\mu \chi_{\vec{e}} \leq 1$$

and, by (x),

$$\sum_{j=1}^{3} \mathbb{E}_\mu \chi_{\vec{e}_j} = 1.$$

(xi)

If $d\mu(\omega)$ is given by a Dirac δ - function on Ω then

$$\varphi(\vec{e}) := \mathbb{E}_{\mu=\delta}(\chi_{\vec{e}}) = 0 \text{ or } 1, \quad (xii)$$

for all $\vec{e}$, with $\sum_{j=1}^{3} \mathbb{E}_{\mu} \chi_{\vec{e}_j} = 1, \quad (xiii)$

for arbitrary orthonormal dreibeins $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. Functions φ with the above properties do not exist!

Proof 1: K - S

Proof 2: Use Gleason's thm.:

$$\varphi : \vec{e} \in S^2 \longmapsto \varphi(\vec{e}) \quad (= 0 \text{ or } 1, \forall \vec{e})$$

After complexification: φ is an additive measure (see (xiii)!) on the lattice of projections acting on $\mathbb{C}^3$.

$\overset{\text{Gleason}}{\Longrightarrow}$ $\exists$ a s.a. matrix $X \geq 0$ with

$\mathrm{tr}\, X = 1$ such that

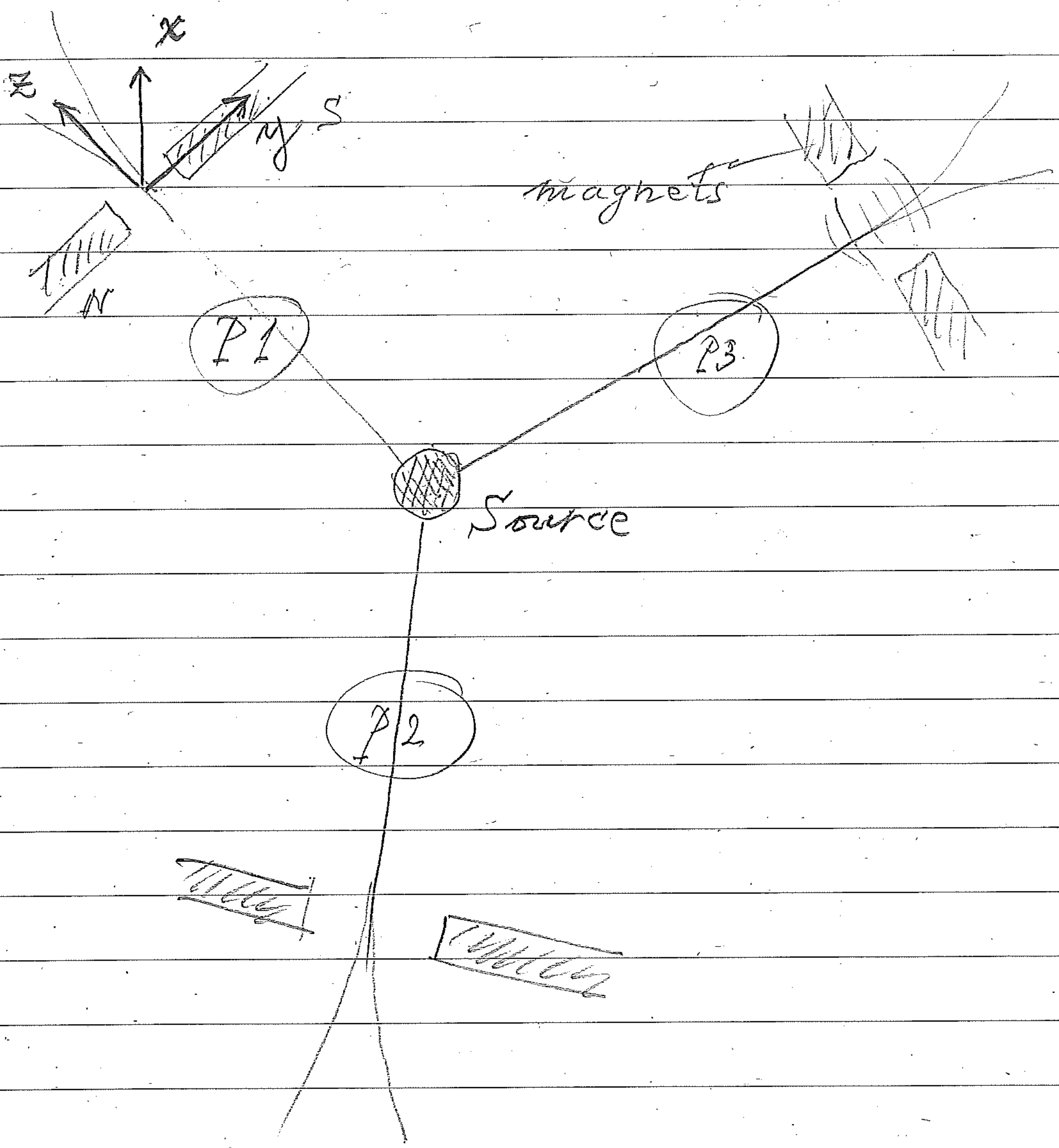
$$\varphi(\vec{e}) = \mathrm{tr}\,(X P(\vec{e})) = (\vec{e}, X\vec{e})$$

$\Rightarrow$ For some $\vec{e}$, $0 < \varphi(\vec{e}) < 1$,

contradicting (xii)!

Compare to Kakutani's thm.; ($\nearrow$ Dyson).

3.2 EPR & Bell



Q_1, Q_2, Q_3 as in 4.1 ; $\Big(Q_1 = \underset{\sigma_x \otimes 1 \otimes 1}{A_1} \, \underset{1 \otimes \sigma_y \otimes 1}{B_2} \, B_3$

$$\Psi_1 = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle \otimes |\uparrow\rangle \otimes |\uparrow\rangle - |\downarrow\rangle \otimes |\downarrow\rangle \otimes |\downarrow\rangle\right]$$

Source produces 3 spin-½ particles in state Ψ_1 (repeatedly!).

<u>EPR notions of "reality" and "locality"</u>

(R) Quantities (of a system) whose values can be predicted with <u>certainty</u> are "elements of reality".

(L) Elements of reality of a system, S_1, cannot be influenced by measurements at another sytem, S_2, space-like separated from S_1.

For Ψ_1:

$$Q_1 \Psi_1 = Q_2 \Psi_1 = Q_3 \Psi_1 = \Psi_1$$

$\longrightarrow$ Measure σ_x for P1, σ_y for P2

$\Rightarrow$ σ_y for P3 determined $\longrightarrow$ hence

σ_y an "element of reality."

Measure σ_y for P1, σ_y for P2

$\Rightarrow$ σ_x for P3 determined $\longrightarrow$ hence

σ_x an "element of reality."

> Since $[\sigma_x, \sigma_y] = \sigma_z \neq 0 \Rightarrow \sigma_x$ and σ_y for P3 <u>cannot</u> simultaneously be "elements of reality."

<u>Problem:</u> (L) not valid in strict form!

Bell's analysis:

Assume local hidden variables theory.

$\Rightarrow \exists$ random variables $a_i(\omega) := f_{A_i}(\omega)$, $b_i(\omega) := f_{B_i}(\omega)$, $i = 1, 2, 3$, such that

$$1 = \langle \Psi_1, \underbrace{A_1 B_2 B_3}_{Q_1} \Psi_1 \rangle = \int_{\Omega} a_1(\omega) b_2(\omega) b_3(\omega) \, d\rho_{[\Psi_1]}(\omega) \qquad (xiv)$$

$$\left.\begin{aligned} 1 &= f_1 = f_{A_i^2} = f_{A_i}^2 = a_i^2 \\ 1 &= f_1 = f_{B_i^2} = f_{B_i}^2 = b_i^2 \end{aligned}\right\} \forall i$$

$\Rightarrow a_i(\omega) = \pm 1$, $b_i(\omega) = \pm 1$, $\forall \omega, \forall i$.

$\Rightarrow$ By (xiv),

$$a_1 b_2 b_3 = 1, \text{ a.e.}, \; + \text{cycl.} \qquad (xv)$$

on $\operatorname{supp} d\rho_{[\Psi_1]}$.

$(**)$ + cycl. $\Longrightarrow$

$$a_1 b_2 b_3 \, b_1 a_2 b_3 \, b_1 b_2 a_3 = a_1 a_2 a_3 = 1$$

$$\Rightarrow \quad a_1 a_2 a_3 = 1 \,, \quad \text{on supp } d\rho_{[\Psi_1]}$$

On the other hand,

$$1 = \langle \Psi_1, Q_1 Q_2 Q_3 \Psi_1 \rangle = \langle \Psi_1, -A_1 A_2 A_3 \Psi_1 \rangle$$

$$= -\int a_1(\omega)\, a_2(\omega)\, a_3(\omega)\, d\rho_{[\Psi_1]}(\omega)$$

$$\Rightarrow \quad a_1 a_2 a_3 = -1 \,, \quad \text{on supp } d\rho_{[\Psi_1]}$$

$\rightarrow$ $\boxed{\text{Contradiction}}$