4. Information Loss, Events, Direct/ Projective/von Neumann Measurements, ETH Ontology

This follows mostly:

- My slides
- Quantum Probability Theory, ... (with B. Schubnel)
- A "Garden of Forking Paths" – ... (with Ph. Blanchard & B. Schubnel)

Contents.

4.1. A formulation of Quantum Theory in terms of time-localized operator algebras

4.2. States of physical interest; passage to von <u>Neumann algebras</u>

4.3. Time evolution (Buchholz & Roberts); Information Loss

4.4. Instruments

4.5. Digression on Tomita–Takesaki theory, and Conditional Expectations

4.6. Centralizers of states on von Neumann algebras and their centers – <u>Events</u>, and their observations using instruments.

$$\boxed{\mathcal{C}_{\rho_1 \oplus \rho_2} \neq \mathcal{C}_{\rho_1} \vee \mathcal{C}_{\rho_2}} \quad !$$

4.7. ETH ontology.

4.8. Toy examples of ETH ontology based on Lindblad evolutions and their unraveling.

4.1. A formulation of Quantum Theory in terms of time-localized operator algebras:

Consider NR quantum theory, and assume $\exists$ global time modeled by $\mathbb{R}$.

$\mathbb{R} \supset I \longrightarrow \mathcal{E}_I$; a C^*-algebra with $\mathbb{1}$.

Assumption:

$$I \subseteq I' \Rightarrow \mathcal{E}_I \subseteq \mathcal{E}_{I'} \tag{4.1}$$

Def.

$$\mathcal{E}_{\geq t} := \overline{\bigvee_{I \subset [t,\infty)} \mathcal{E}_I}^{\|\cdot\|} \tag{4.2}$$

$$\mathcal{E} := \overline{\bigvee_{t > -\infty} \mathcal{E}_{\geq t}}^{\|\cdot\|} \tag{4.3}$$

Then

$$\mathcal{E} \supseteq \mathcal{E}_{\geq t} \supseteq \mathcal{E}_I , \tag{4.4}$$

for arb $I \subseteq [t, \infty)$, arb. $t \in \mathbb{R}$.

[Phys. interpretation of $\mathcal{E}_I, \mathcal{E}_{\geq t}, \ldots$: As in older presentations!]

States = states on C^*-algebra $\mathcal{E}$.

4.2 States of physical interest, ...

$\mathcal{S}$: stratum of "states of physical interest" on $\mathcal{E}$.

Assume $\exists$ state $\omega_{ref} \in \mathcal{S}$ such that all other states in $\mathcal{S}$ are normal w.r. to ω_{ref}.

Apply GNS construction to $(\mathcal{E}, \omega_{ref})$:

$\xrightarrow[GNS]{}$ $\exists$ Hilbert space $\mathcal{H}$, *rep., π, of $\mathcal{E}$ on $\mathcal{H}$, unit ray $\Omega \in \mathcal{H}$ such that

$$\omega_{ref}(A) = \langle \Omega, \pi(A)\Omega \rangle_{\mathcal{H}}, \quad \forall A \in \mathcal{E}$$

From now on, will work on $\mathcal{H}$ and pass to von Neumann algebras

$$\overline{\mathcal{E}_I}, \quad \overline{\mathcal{E}_{\geq t}}, \quad \overline{\mathcal{E}}, \qquad (4.5)$$

where $^-$ denotes the closure in the (ultra-)weak topology on $\mathcal{B}(\mathcal{H})$. We will occasionally omit "$-$".

4.3. Autonomous systems

System autonomous iff, for all $t \geq 0$,

$$\tau_t : \mathcal{E}_{\geq t'} \longrightarrow \mathcal{E}_{\geq t'+t} \subseteq \mathcal{E}_{\geq t'} \qquad (4.6)$$

$\{\tau_t\}_{t \geq 0}$: * endomorphisms of $\mathcal{E}^-_{\geq t'}$, $\forall t'$.

Assume that

$$t \longmapsto \omega_{ref}\left(A^* \tau_t(B)\right) \text{ cont. in } t,$$

$\forall A, B \in \mathcal{E}_{\geq t'}$, all t'

[At zero temperature,

$$t \longmapsto \omega_0\left(A^* \tau_t(B)\right)$$

(vacuum)

extends to analytic function on

$\mathbb{C}_+ = \{t \mid \operatorname{Im} t > 0\}$, bd. by

$$\sqrt{\omega_0(A^*A)\, \omega_0(B^*B)},$$

$\forall A, B \in \mathcal{E}_{\geq t'}$, $\forall t'$.]

Theorem.

For autonomous systems we have the alternative.

(i) $\mathcal{E}^-_{\geq t}$ indep. of t and of type I

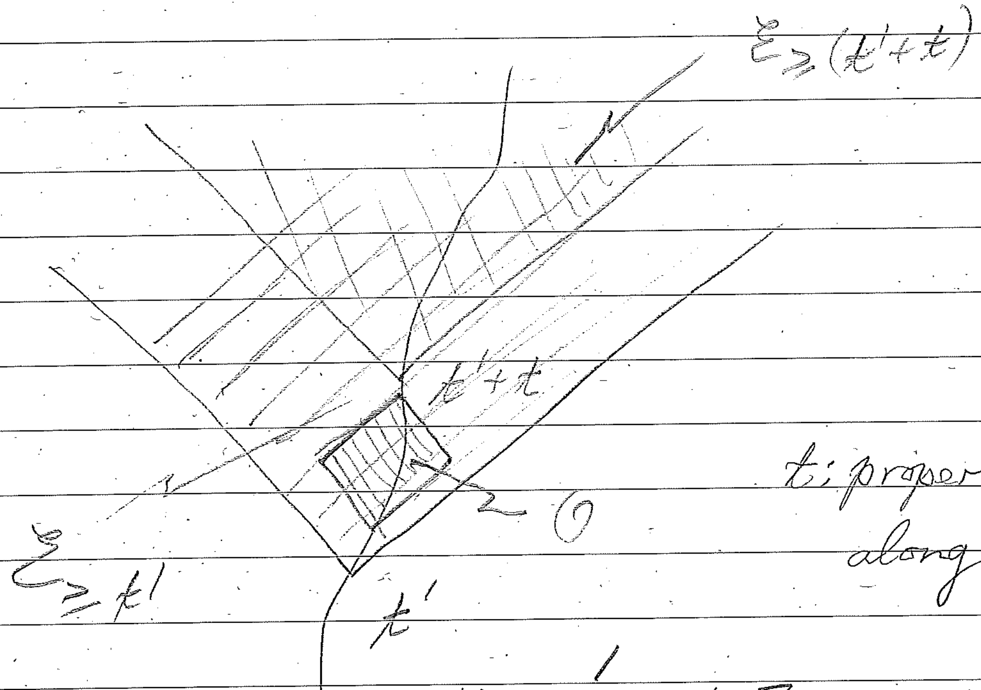
or

(ii) $\mathcal{E}^-_{\geq t} \subsetneq \mathcal{E}^-_{\geq (t+s)}$, $s>0$, and

then

$$\mathcal{E}^-_{\geq t} \text{ of type } III.$$

Information Loss in QED:



t: proper time along $\mathcal{L}$

$$\mathcal{E}'_{\geq(t'+t)} \cap \mathcal{E}^-_{\geq t'} \supseteq \mathcal{A}''_{asy}(\mathcal{O})$$

↪ Idea of Information Loss:

"Elemental states":

ω state on $\mathcal{E}_{\geq t}$ "elemental" iff $\overline{\mathcal{E}_{\geq t}}^{w}$ in GNS rep. assoc. with $(\mathcal{E}_{\geq t}, \omega)$ is a factor of type III_1.

↪ Decomposition of states of physical interest into classes corresp. to "elemental" states.

$$\boxed{\begin{aligned}&\text{Class of elemental state } \omega\\ &= \{\omega \circ \gamma \mid \gamma \in \mathrm{Inner}\, \mathcal{E}_{\geq t}\}\end{aligned}} \tag{4.7}$$

Information Loss:

$$\boxed{\mathcal{E}^{-} \supsetneq \mathcal{E}^{-}_{\geq t} \supsetneq \mathcal{E}^{-}_{\geq s}\,, \quad s > t} \tag{4.8}$$

Information Loss in non-autonomous systems.

$$\mathcal{E}_{\geq n} = \mathcal{E}_P \otimes \left(\bigotimes_{j \geq n} \mathcal{A}_j \right)$$

$$\mathcal{E}'_{\geq n} \cap \mathcal{E}_{\geq m} \underset{m<n}{=} \bigotimes_{j=m}^{n-1} \mathcal{A}_j \qquad (4.9)$$

$\hookrightarrow$ $\boxed{\text{Info Loss}}$

4.4 Instruments

An instrument, $\mathcal{I}$, is an abelian C^*-algebra, $(\mathcal{O}_\mathcal{I}$, w. $\mathbb{1}_\mathcal{I})$, along with a family $\{\pi_t\}_{t \in \mathbb{R}}$ of *homomorphisms,

$$\pi_t : \mathcal{O}_\mathcal{I} \longrightarrow \mathcal{E}_{\geq t}, \qquad (4.10)$$

called reps. of $\mathcal{I}$ in $\mathcal{E}_{\geq t}$.

(Generalization: $\mathcal{E}_{\geq t} \longrightarrow \mathcal{E}_I$, $I \subset \mathbb{R}, \dots$)

$\longrightarrow$ Apply Gel'fand isomorphism to $\mathcal{O}_\mathcal{I}$!

To simplify matters, introduce

Assumption.

All instruments $\mathcal{J}$ are finite, meaning

$\mathcal{O}_{\mathcal{J}}$ is finite-dim. $\Longrightarrow$

$$\mathcal{O}_{\mathcal{J}} = \left\langle \Pi_{\xi} \mid \xi \in X, \, |X| < \infty \right\rangle, \qquad (4.11)$$

$$\left.\begin{aligned} &\Pi_{\xi} \Pi_{\xi'} = \delta_{\xi\xi'} \Pi_{\xi'}, \quad \Pi_{\xi}^{*} = \Pi_{\xi}, \\ &\sum_{\xi \in X} \Pi_{\xi} = \mathbb{1}_{\mathcal{O}_{\mathcal{J}}}. \end{aligned}\right\} \qquad (4.12)$$

Instruments will serve to describe

measurements and observations.

4.5. Digression on Tomita-Takesaki theory and Conditional Expectations

(see Bratteli & Robinson, vol. I!)

$\mathfrak{M}$: von Neumann algebra acting on $\mathcal{H}$

$\mathfrak{M}'$: commutant of $\mathfrak{M}$

$\mathfrak{M}'' \equiv (\mathfrak{M}')' \underset{\uparrow}{=} \mathfrak{M}$

for von Neumann alg.

$Z_{\mathfrak{M}} := \text{center of } \mathfrak{M} = \mathfrak{M}' \cap \mathfrak{M}$

Def.

(i) $\mathfrak{M}$ a factor iff $Z_{\mathfrak{M}} = \{\mathbb{C}\mathbb{1}\}$.

(ii) $\mathfrak{M}$ σ-finite iff every collection of mutually orthogonal projections in $\mathfrak{M}$ is finite our countably infinite.

(iii) $\psi \in \mathcal{H}$ separating iff $A\psi = 0$, $A \in \mathfrak{M}$, implies $A = 0$

$\psi \in \mathcal{H}$ cyclic iff $\{A\psi \mid A \in \mathfrak{M}\}$ is dense in $\mathcal{H}$.

Lemma.

$$\left.\begin{array}{l}\psi \text{ cyclic for } \mathfrak{M} \\ \text{separating}\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l}\psi \text{ separating for } \mathfrak{M}' \\ \text{cyclic}\end{array}\right. \qquad (4.13)$$

Proof: Obvious.

Generalization of separating: ω state on $\mathfrak{M}$

faithful iff $\omega(A) > 0$, for all $0 \neq A \in \mathfrak{M}_+$

Lemma.

$\mathfrak{M}$ is σ-finite $\Longleftrightarrow$ $\exists$ a cyclic and separating vector in a normal rep. π of $\mathfrak{M}$

Def. A closed op. A on $\mathcal{H}$ affiliated w. $\mathfrak{M}$ ($A \eta \mathfrak{M}$) iff

$$\mathfrak{M}' D(A) \subseteq D(A) \quad \text{and}$$

$$AA' \supseteq A'A, \quad \forall A' \in \mathfrak{M}'$$

Here one may want to digress into Haag-Hugenholtz-Winnink!

<u>Lemma.</u> A closed, $A \eta \mathfrak{M}$.

$A = U|A|$: polar dec. of A

Then U and spect. projections of $|A|$ belong to $\mathfrak{M}$.

<u>The main characters of the drama.</u>

Let $\Omega \in \mathcal{H}$ be cyclic and separating for $\mathfrak{M}$

[$\Leftrightarrow$ Ω cyclic and separating for $\mathfrak{M}'$;

$\{A\Omega \mid A \in \mathfrak{M}\}$ <u>dense</u> in $\mathcal{H}$,

$A\Omega = 0,\ A \in \mathfrak{M} \Rightarrow A = 0$.]

<u>Def.</u>

$$\left.\begin{aligned} S_0 A \Omega &:= A^* \Omega, \quad A \in \mathfrak{M} \\ F_0 B \Omega &:= B^* \Omega, \quad B \in \mathfrak{M}' \end{aligned}\right\} \quad (4.14)$$

S_0, F_0 <u>are densely defined anti-linear operators</u> on $\mathcal{H}$.

Proposition.

- S_0 and F_0 are closable.
- $S_0^* = F_0^-$, $F_0^* = S_0^-$,

where the bar indicates closure.

- Given $\psi \in D(S_0^-)$, $\exists$ $Q \eta \mathfrak{M}$ such that

$$\psi = Q\Omega \ , \quad S_0^- \psi = Q^* \Omega \ ;$$

and likewise for F_0^-.

Sketch of proof. $A \in \mathfrak{M}$, $B \in \mathfrak{M}'$

$$\langle B\Omega, S_0 A\Omega \rangle = \langle B\Omega, A^*\Omega \rangle$$

$$\underset{[A,B]=0}{=} \langle A\Omega, B^*\Omega \rangle$$

$$= \langle A\Omega, F_0 B\Omega \rangle$$

$\Rightarrow F_0 \subseteq S_0^* \Rightarrow S_0^*$ densely defined; S_0 closable.

- - - - - -

(B-R, vol. I, page 88)

<u>Def.</u> $S := \overline{S_0}$, $F = \overline{F_0}$.

<u>Polar decomposition:</u>

$$S = J \Delta^{1/2}, \qquad (4.15)$$

$\Delta = \Delta^* \geq 0$, J <u>anti-unitary.</u>

$\uparrow$ modular op. $\quad \uparrow$ modular conjugation

assoc. w. $(\mathcal{M}, \Omega)$.

<u>Proposition.</u>

$\Delta = S^* S = FS$, $\Delta^{-1} = SF$

$S = J\Delta^{1/2}$, $F = J\Delta^{-1/2}$

$J = J^*$, $J^2 = \mathbb{1}$ (involution)

$$\Delta^{-1/2} = J \Delta^{1/2} J$$

<u>Remarks.</u> $S_0^{\,2} A\Omega = S_0 (S_0 A \Omega) = S_0 (A^* \Omega) = A\Omega$

$$\Rightarrow S_0^{\,2} = \mathbb{1} \Rightarrow S^2 = \mathbb{1}$$

$$\Rightarrow J\Delta^{1/2} = S = S^{-1} = \Delta^{-1/2} J^*$$

4.16

$$\Rightarrow \quad J^2 \Delta^{1/2} = \underbrace{J \Delta^{-1/2} J^*}_{\geq 0\ !} \qquad *$$

Uniqueness of polar decomposition $\Rightarrow$

$$J^2 = \mathbb{1}$$

With $J^* J = \mathbb{1} = J^2 \Rightarrow J^* = J$

With $* \Rightarrow$ $\boxed{\Delta^{\mp 1/2} = J \Delta^{\pm 1/2} J}$ (4.16)

Next, for A, B, C in $\mathfrak{M}$,

$$(SAS)\, BC\Omega = SAC^* B^* \Omega$$

$$= BCA^* \Omega$$

and

$$B(SAS)\, C\Omega = BSAC^* \Omega$$

$$= BCA^* \Omega$$

Since $\{C\Omega \mid C \in \mathfrak{M}\} \subseteq \mathcal{H}$,

$$[SAS, B] = 0, \quad \text{for arb. } B \in \mathfrak{M}$$

$$\Rightarrow \quad SAS \,\eta\, \mathfrak{M}', \quad \forall A \in \mathfrak{M}.$$

If Δ (and hence S, F, Δ^{-1}) were

bounded then

$$S\mathcal{M}S \subseteq \mathcal{M}', \quad F\mathcal{M}'F \subseteq \mathcal{M}$$

$$\Longrightarrow \Delta\mathcal{M}\Delta^{-1} \subseteq \mathcal{M} \qquad (\text{easy};$$

$$\Delta\mathcal{M}\Delta^{-1} = \Delta^{1/2} J J \Delta^{1/2} \mathcal{M} \Delta^{-1/2} J J \Delta^{-1/2}$$

$$= FS\mathcal{M}SF \subseteq F\mathcal{M}'F \subseteq \mathcal{M}$$

Likewise for all powers of Δ

$$\Rightarrow \boxed{\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}, \quad \forall t \in \mathbb{R}.} \tag{4.17}$$

$$\Rightarrow J\mathcal{M}J = J\underbrace{\Delta^{1/2}\mathcal{M}\Delta^{-1/2}}_{=\mathcal{M}}J$$

$$= S\mathcal{M}S \subseteq \mathcal{M}'$$

$$J\mathcal{M}'J = J\underbrace{\Delta^{-1/2}\mathcal{M}'\Delta^{1/2}}_{=\mathcal{M}'}J = F\mathcal{M}'F \subseteq \mathcal{M}$$

$$\Rightarrow \boxed{J\mathcal{M}J = \mathcal{M}'} \tag{4.18}$$

Now remove ass. # ! (See B-R, vol. I)

KMS condition.

ω : faithful, normal state on $\mathfrak{M}$.

$(\mathcal{H}_\omega, \pi_\omega, \Omega)$ obtained by GNS constr.

from $(\mathfrak{M}, \omega)$.

Since ω faithful $\Rightarrow$ Ω (cyclic &) separating.

$$\sigma_t^\omega(A) := \pi_\omega^{-1}\left(\Delta^{it}\pi_\omega(A)\Delta^{-it}\right)$$

$\uparrow$

modular *automorphism group assoc. w. $(\mathfrak{M}, \omega)$.

KMS condition w. $\beta = 1$:

$$\boxed{\begin{aligned} \omega\left(\sigma_i(A)B\right) &= \omega\left(\sigma_{i/2}(A)\sigma_{-i/2}(B)\right) \\ &= \omega(BA) \end{aligned}} \quad (4.19)$$

By calculation in GNS rep.:

$$\omega\left(\sigma_{i/2}(A)\sigma_{-i/2}(B)\right) = \langle \Delta^{1/2}A^*\Omega, \Delta^{1/2}B\Omega\rangle$$

$$= \langle J(J\Delta^{1/2}A^*)\Omega, J(J\Delta^{1/2}B)\Omega \rangle$$

$$= \langle JSA^*\Omega, JSB\Omega \rangle$$

$$= \langle JA\Omega, JB^*\Omega \rangle$$

$$= \langle B^*\Omega, A\Omega \rangle$$

$$= \langle \Omega, BA\Omega \rangle$$

$$= \omega(BA).$$

Conditional Expectations; (employs T-T th.!)
(see Takesaki, vol. I, p. 332; vol. II, p. 211)

Let ω be a faithful, normal state on a von Neumann algebra $\mathfrak{M}$. (A state is automatically a semi-finite weight! See p. 41, vol. II)

Let $\mathfrak{N} \subset \mathfrak{M}$ a von Neumann subalgebra of $\mathfrak{M}$.

Def.

A linear map $\in : \mathfrak{M} \xrightarrow[\text{onto}]{} \mathfrak{N}$

is called a conditional expectation of $\mathfrak{M}$

onto $\mathfrak{N}$ with respect to ω iff

(i) $\|\in(X)\| \leq \|X\|, \quad \forall X \in \mathfrak{M}$

(ii) $\in(X) = X, \quad \forall X \in \mathfrak{N}$

(iii) $\omega = \omega \circ \in$

Lemma. (See Takesaki, vol. II, p. 211; ~~vol. III~~)

- $\in(X^*X) \geq 0, \quad \forall X \in \mathfrak{M}$
- $\in(AXB) = A\in(X)B, \quad \forall A,B \in \mathfrak{N}, \forall X \in \mathfrak{M}$
- $\in(X)^* \in(X) \leq \in(X^*X), \quad \forall X \in \mathfrak{M}$
- $\in$ is completely positive ($\rightarrow$ Stinespring)

Theorem. (See Takesaki, vol. II, p. 211)

$\mathfrak{M}, \mathfrak{N}, \omega$ as in above Def., Then

Existence of a conditional expectation
$\mathcal{E} : \mathcal{M} \to \mathcal{N}$ w. r. to ω
$\Longleftrightarrow \quad \sigma_t^{\omega}(\mathcal{N}) = \mathcal{N}, \quad \forall t \in \mathbb{R}$ (4.20)

modular automorphism group assoc. w. $(\mathcal{M}, \omega)$.

[$\mathcal{E}$ is normal and uniquely determined by ω.]