Digression on:

Classification of von Neumann algebras

Preliminaries. (See B-R, vol. I, Sect. 2.7)

1. **Def.** A von Neumann algebra $\mathfrak{M}$ is said to be semi-finite iff $\exists$ a faithful, normal semi-finite trace, τ, on $\mathfrak{M}$.

an example of a "weight".

2. **Def.** ω a weight on $\mathfrak{M}$; ($\nearrow$ Prop. 2.7.9, ff)

$$\mathfrak{M}_{\omega +} := \{ A \in \mathfrak{M}_+ \mid \omega(A) < \infty \}$$

$$\mathcal{L}_\omega := \{ A \in \mathfrak{M} \mid \omega(A^*A) < \infty \}$$

$$\mathfrak{M}_\omega := \langle \mathfrak{M}_{\omega +} \rangle \equiv \text{complex, linear span of } \mathfrak{M}_\omega$$

Then $\mathfrak{M}_\omega = \mathcal{L}_\omega^* \mathcal{L}_\omega$

3. **Def.** A weight ω is called

- normal iff

$$\omega(A) = \sum_i \omega(A_i),$$

for every sequence $\{A_i\} \subset \mathfrak{M}_+$ with

$$\sum_i A_i =: A \in \mathfrak{M}_+ .$$

- ω is called <u>faithful</u> iff

$$A \in \mathfrak{M},\ \omega(A) = 0 \implies A = 0$$

- ω is called <u>semi-finite</u> iff

$\mathfrak{M}_\omega$ is σ-weakly dense in $\mathfrak{M}$.

<u>Prop.</u> <u>Every von Neumann alg. admits a normal, faithful, semi-finite weight.</u>

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4. <u>Connes' Radon-Nikodym Theorem</u>

ω, ρ: normal, faithful, semi-finite weights (<u>states</u>) on $\mathfrak{M}$. Then there exists a <u>cocycle</u> $\Gamma_t \equiv (D\omega : D\rho)_t$ such that

$$(1)\quad \sigma_t^\omega(A) = \Gamma_t\, \sigma_t^\rho(A)\, \Gamma_t^*$$

$$(2)\quad \Gamma_{t+s} = \Gamma_t\, \sigma_t^\rho(\Gamma_s)$$

(3) $\Gamma_t^* = (D\rho : D\omega)_t$,

$$(D\omega : D\rho)_t (D\rho : D\varphi)_t = (D\omega : D\varphi)_t$$

(4) $\omega(A) = \rho(UAU^*)$, $\mathcal{M} \ni U$ unitary

$$\Leftrightarrow \Gamma_t = (D\omega : D\rho)_t = U^* \sigma_t^\rho(U)$$

Classification of factors; (i.e., $\mathcal{M}$ w. $Z(\mathcal{M}) = \{\mathbb{C}\mathbb{1}\}$).

I. $\mathcal{M}$ factor of type I $\Leftrightarrow$ $\mathcal{M} = B(\mathcal{H})$

dim $\mathcal{H}$ is then a complete invariant for $\mathcal{M}$.

II $\mathcal{M}$ factor of type II $\Leftrightarrow$ $\mathcal{M}$ is semi finite,

but not of type I

Type II_1 : Trace is finite $\Rightarrow$ i.e., a state

Type II_∞ : " " infinite.

III $\mathcal{M}$ is of type III if it is not semi-finite

Infinite von Neumann algebras are **crossed products** of semi-finite von Neumann algebras by a σ-weakly continuous one-parameter group, $\{\alpha_t\}_{t\in\mathbb{R}}$ of *automorphisms; if τ is the trace on the semi-finite v.N. algebra then

$$\tau\circ\alpha_t = e^{-t}\tau, \quad t\in\mathbb{R}.$$

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Def. If $\mathcal{M}$ is a factor define

$$S(\mathcal{M}) = \bigcap_{\omega} \sigma(\Delta_\omega),$$

ω: normal semifinite weight on $\mathcal{M}$.

$\sigma(A)$: spectrum of $A = A^*$.

Remark. Clearly $S(\mathcal{M}) \subseteq \mathbb{R}_+$.

<u>Theorem</u>. (Connes)

$$T_\omega := \{ t \in \mathbb{R} \mid \sigma^\omega_t \text{ inner} \}$$

Then T_ω is an additive subgroup of $\mathbb{R}$ <u>indep.</u> of ω. If $\mathcal{M}$ is a factor define

$$\Gamma(\mathcal{M}) := \{ \lambda \in \mathbb{R} \mid t \in T \Rightarrow e^{i\lambda t} = 1 \}$$

Then

$$\Gamma(\mathcal{M}) = \log \left(S(\mathcal{M}) \setminus \{0\} \right)$$

$\Longrightarrow$ $S(\mathcal{M}) \setminus \{0\}$ is a <u>closed subgroup</u> of the multiplicative group of positive real numbers. $\Longrightarrow$

(i) $S(\mathcal{M}) = \{1\}$

(ii) $S(\mathcal{M}) = [0, \infty)$

(iii) $S(\mathcal{M}) = \{0\} \cup \{\lambda^n\}_{n \in \mathbb{Z}}$, $\lambda \in (0,1)$.

(iv) $S(\mathcal{M}) = \{0, 1\}$

(i) $\iff$ $\mathfrak{M}$ of type I or of type II

(namely $\mathfrak{M}$ semi finite)

(ii) – (iv) $\iff$ $\mathfrak{M}$ of type III

We say that

$\mathfrak{M}$ of type III_0 $\iff$ (iv)

$\mathfrak{M}$ of type III_λ $\iff$ (iii)

$\mathfrak{M}$ of type III_1 $\iff$ (ii)

Type III_1 factor is crossed product of type II_∞ factor by some $\{\alpha_t\}_{t\in\mathbb{R}}$.

A factor $\mathfrak{M}$ is called <u>hyperfinite</u> iff $\mathfrak{M}$ is generated by an ascending sequence of finite-dimensional matrix algebras.

If $\mathfrak{M}$ is hyperfinite $\Rightarrow$ $\exists$ projection E

$E : \mathcal{B}(\mathcal{H}) \to \mathfrak{M}$,

$E^2 = E, \quad E\,\mathcal{B}(\mathcal{H}) = \mathfrak{M}, \quad \|E\| = 1,$

which is a <u>conditional expectation</u>.

Converse is true, too; except possibly if $\mathfrak{M}$ is type III_1 factor.

Hyperfinite factors of type II_1, type II_∞, and type III_λ, $0 < \lambda \leq 1$ are <u>unique</u>.

Type III_0 classified by flow $\{Z_{\mathfrak{N}}, \alpha\}$, where

$$\mathfrak{M} = \mathfrak{N} \rtimes \alpha,$$

4.6. Centralizers of States on von Neumann Algebras and Their Centers – Events and Their Observations Using Instruments

$\mathfrak{M}$; a von Neumann algebra

ω; a state on $\mathfrak{M}$

<u>Def.</u> The <u>centralizer</u> of ω in $\mathfrak{M}$ is the von Neumann subalgebra, $\mathcal{C}_\omega$, of $\mathfrak{M}$ defined by

$$\mathcal{C}_\omega := \{X \in \mathfrak{M} \mid \mathrm{ad}_X\, \omega = 0\}, \qquad (4.21)$$

where $\mathrm{ad}_X\, \omega$ is the bounded linear functional on $\mathfrak{M}$ defined by

$$\mathrm{ad}_X\, \omega(Y) := \omega([X, Y]) \qquad (4.22)$$

<u>Lemma.</u> $\mathcal{C}_\omega$ is a von Neumann subalgebra of $\mathfrak{M}$.

<u>Proof.</u> (i) Clearly, if $X_1, X_2 \in \mathcal{C}_\omega$, $\lambda_1, \lambda_2 \in \mathbb{C}$ then $\lambda_1 X_1 + \lambda_2 X_2 \in \mathcal{C}_\omega$, by linearity of ω.

(ii) If $X \in \mathcal{C}_\omega$ then $X^* \in \mathcal{C}_\omega$:

$$
\begin{aligned}
\omega([X^*, y]) &= \langle \Omega, (X^* y - y X^*) \Omega \rangle \\
&= \overline{\langle (y^* X - X y^*) \Omega, \Omega \rangle} \\
&= \langle \Omega, (y^* X - X y^*) \Omega \rangle \\
&= -\, \omega([X, y^*]) \\
&= -\, ad_X\, \omega(y^*) \\
&= 0, \quad \forall\, y \in \mathcal{M} \text{ if } X \in \mathcal{C}_\omega .
\end{aligned}
$$

$\Rightarrow ad_{X^*}\, \omega = 0$.

(iii) $X_1, X_2 \in \mathcal{C}_\omega \Rightarrow X_1 \cdot X_2 \in \mathcal{C}_\omega$:

$$
\begin{aligned}
ad_{X_1 \cdot X_2}\, \omega(y) &= \omega([X_1 X_2, y]) \\
&= \omega(X_1 (X_2 y) - (X_2 y) X_1) \\
&\quad + \omega(X_2 (y X_1)) - \omega((y X_1) X_2)
\end{aligned}
$$

$$= \mathrm{ad}_{X_1}\omega(X_2 y) + \mathrm{ad}_{X_2}\omega(y X_1)$$

$$= 0, \quad \forall y \in \mathfrak{M}.$$

<u>Observation</u>. If $X \in \mathcal{C}_\omega$, $X = X^*$, then

$$\mathrm{Ad}_{e^{itX}}\,\omega = \omega, \quad \forall t$$

Here

$$\mathrm{Ad}_{e^{itX}}\,\omega(y) := \omega(e^{itX}\, y\, e^{-itX});$$

and conversely.

<u>Theorem</u>. Let ω be <u>faithful</u> state on $\mathfrak{M}$.
Let σ_t^ω be the modular automorphism group on $\mathfrak{M}$ assoc. with $(\mathfrak{M}, \omega)$.

Then

$$\mathcal{C}_\omega = \{ X \in \mathfrak{M} \mid \sigma_t^\omega(X) = X, \ \forall t \in \mathbb{R} \}$$

Proof. Exercise, using KMS condition (See, e.g., F-Schubnel, "Quantum Prob. Theory…")

Corollary. $\exists$ conditional expectation

$$\overline{E}_\omega : \mathfrak{M} \to \mathcal{C}_\omega,$$

with $\omega = \omega \circ \overline{E}_\omega$.

Important observation:

$$\boxed{\omega \text{ is a normalized trace on } \mathcal{C}_\omega} \tag{4.23}$$

$$\Rightarrow \mathcal{C}_\omega = \begin{cases} \text{abelian von Neumann alg.} \\ M_n(\mathbb{C}), \quad n \in \mathbb{N} \\ \text{type } \mathrm{II}_1 \text{ von Neumann alg.} \end{cases} \tag{4.24}$$

or direct sum of such algebras

$\mathcal{C}_\omega = \bigoplus_i M_{n_i}(\mathbb{C}) \Leftrightarrow \mathfrak{M}$ type I v.N. alg.

$\mathfrak{M}$ type III $\Rightarrow \mathcal{C}_\omega$ type II_1.

If $\mathcal{C}_\omega$ is <u>not</u> a factor (e.g.,
$\mathcal{C}_\omega = \bigoplus_i M_{n_i}(\mathbb{C})$, non-trivial direct
sum of matrix algebras) then $\mathcal{C}_\omega$
contains a <u>non-trivial center</u>

$$\left.\begin{array}{l} Z_\omega := \text{center of } \mathcal{C}_\omega \\ \epsilon_\omega : \text{conditional expectation } \mathcal{M} \to Z_\omega \end{array}\right\}$$

Return to analysis of <u>isolated physical</u>
<u>systems</u>: S a phys. system described
by

- $\{\mathcal{E}_I\}_{I \subset \mathbb{R}}$: net of algebras indexed
  by intervals $I \subset \mathbb{R}$ of proper time
- $\mathcal{S}$: stratum of states on $\{\mathcal{E}_I\}_{I \subset \mathbb{R}}$
  of physical interest
- $\mathcal{O}_S := \{\mathcal{J}_j \mid j \in \text{Inst.}\}$: family of
  instruments.

<u>Def.</u> <u>Events</u> happening at time t, given state, ρ_t, of Σ immediately before time t: (ρ_t : a normal state on $\mathcal{E}^-_{\geq t}$)

$\mathcal{C}_{\rho_t}$: centralizer of ρ_t on $\mathcal{E}^-_{\geq t}$.

$\mathcal{Z}_{\rho_t}$: center of $\mathcal{C}_{\rho_t}$.

An <u>event</u> happening at time t is an orthogonal projection, $\Pi = \Pi^* = \Pi^2$, belonging to $\mathcal{Z}_{\rho_t}$.

Assume that

$$\mathcal{Z}_{\rho_t} = \left\langle \Pi_\xi(t) \,\middle|\, \xi \in \mathcal{X}_t \right\rangle, \tag{4.25}$$

$\mathcal{X}_t$ countable $\left(|\mathcal{X}_t| < \infty\right)$.

Observations (BFS – "Garden…")

Consider von Neumann alg., $\mathfrak{M}$, and a normal state φ on $\mathfrak{M}$. Let $\mathcal{C}_\varphi \subseteq \mathfrak{M}$ be the centralizer of $\mathfrak{M}$. Let $X \in \mathcal{C}_\varphi$, with

$$X = \sum_{\xi \in \mathcal{X}} \xi \, \Pi_\xi \,, \tag{4.26}$$

where $\mathcal{X} \subset \mathbb{R}$, $|\mathcal{X}| =: N < \infty$, ($\mathcal{X}$ inherits order of $\mathbb{R}$).

Since $\mathcal{C}_\varphi$ is a von Neumann subalgebra of $\mathfrak{M}$, it follows that $\Pi_\xi \in \mathcal{C}_\varphi$, $\forall \xi \in \mathcal{X}$.

Thus, for an arbitrary $A \in \mathfrak{M}$,

$$\varphi(A) = \sum_{(\xi,\xi') \in \mathcal{X}\times\mathcal{X}} \varphi\left(\Pi_\xi A \Pi_{\xi'}\right)$$

$$= \sum_{\xi,\xi'} \varphi\left(A \Pi_\xi \delta_{\xi\xi'}\right)$$

$$= \sum_{\xi \in \mathcal{X}} \varphi\left(\Pi_\xi A \Pi_\xi\right) \tag{4.27}$$

Conversely, if, for X as in (4.26), (4.27) holds for all $A \in \mathfrak{M}$ then $X \in \mathcal{C}_\varphi$.

<u>Note.</u> For $X \in Z_\varphi$, (4.27) holds; (this is a special case!).

<u>Lemma.</u> (BFS)

Let X be as in (4.26), and let $\overline{E}_\varphi$ be the conditional expectation: $\mathfrak{M} \rightarrow \mathcal{C}_\varphi$.

Suppose that

$$\| \overline{E}_\varphi (\Pi_\xi) - \Pi_\xi \| < \delta ,$$

$\forall\, \xi \in \mathfrak{X}$. Then

$$\varphi(A) = \sum_{\xi \in \mathfrak{X}} \varphi(\Pi_\xi A \Pi_\xi) + O(\delta \|A\|) \tag{4.28}$$

Proof.

$$\varphi(A) = \sum_{(\xi,\xi')\in\mathcal{X}\times\mathcal{X}} \Big\{ \Big[\varphi(\Pi_\xi A \Pi_{\xi'}) - \varphi(\overline{\epsilon}_\varphi(\Pi_\xi) A \Pi_{\xi'})\Big]$$

$$+ \Big[\varphi(A\Pi_{\xi'}, \overline{\epsilon}(\Pi_\xi)) - \varphi(A\Pi_{\xi'}, \Pi_\xi)\Big]\Big\}$$

$$+ \sum_{\xi\in\mathcal{X}} \Big\{ \Big[\varphi(A\Pi_\xi^2) - \varphi(A\Pi_\xi \overline{\epsilon}_\varphi(\Pi_\xi))\Big]$$

$$+ \Big[\varphi(\overline{\epsilon}_\varphi(\Pi_\xi) A \Pi_\xi) - \varphi(\Pi_\xi A \Pi_\xi)\Big]\Big\}$$

$$+ \sum_{\xi\in\mathcal{X}} \varphi(\Pi_\xi A \Pi_\xi) \qquad (4.29)$$

The proof is completed by using that $\|\overline{\epsilon}_\varphi(\Pi_\xi) - \Pi_\xi\| < \delta$, $\forall \xi \in \mathcal{X}$ and that $\|\Pi_\xi\| = 1$. ∎

Corollary. In above lemma, we can replace $\mathcal{E}_\varphi$ by $\mathcal{Z}_\varphi$ and $\overline{\epsilon}_\varphi$ by ϵ_φ.

<u>Axiom.</u> Assume that, at time $t-$, S is in state ρ_t; and assume that Z_{ρ_t} is as in (4.25), page 4.26. Then

$$\rho_t(A) = \sum_{\xi \in \mathfrak{X}_t} \rho_t\left(\Pi_\xi(t) A \Pi_\xi(t)\right), \tag{4.30}$$

for all $A \in \overline{\mathcal{E}}_{\geq t}$. <u>The axiom says that one of the possible events</u> $\Pi_\xi(t)$, $\xi \in \mathfrak{X}_t$, <u>actually happens</u>; say $\Pi_{\xi_*}(t)$. <u>Then one ought to use the state</u>

$$\rho_t\left(\Pi_{\xi_*}(t)\right)^{-1} \rho_t\left(\Pi_{\xi_*}(t)(\cdot)\Pi_{\xi_*}(t)\right) \tag{4.31}$$

<u>to predict the future of S after time t.</u>

In (4.31),

$$\rho_t\left(\Pi_{\xi_*}(t)\right) \tag{4.32}$$

is the <u>Born probability</u> of the event $\Pi_{\xi_*}(t)$ in the state ρ_t.

The state in (4.31) is henceforth denoted by

$$\boxed{\rho_{t,\xi_*}\ ,\ \xi_* \in \mathfrak{X}_t} \qquad (4.33)$$

The central feature of Quantum Mechanics

If $t' > t$ and $Z_{\rho_{t'}}$ non-trivial then

$$Z_{\rho_{t'}} \underset{\text{i.g.}}{\neq} \bigvee_{\xi \in \mathfrak{X}_t} Z_{\rho_{t',\xi}} \qquad (4.34)$$

Remark. Type I situation:

Suppose that $\rho_{t'} = \Phi(\rho_t)$, $(t'>t)$, where Φ is CP. Consider spect. dec. of ρ_t:

$$\rho_t = \sum_{i=1}^{\infty} p_i(t)\, P_i(t)$$

$$P_i(t) P_j(t) = \delta_{ij} P_i(t) = \delta_{ij} P_i(t)^*.$$

Then

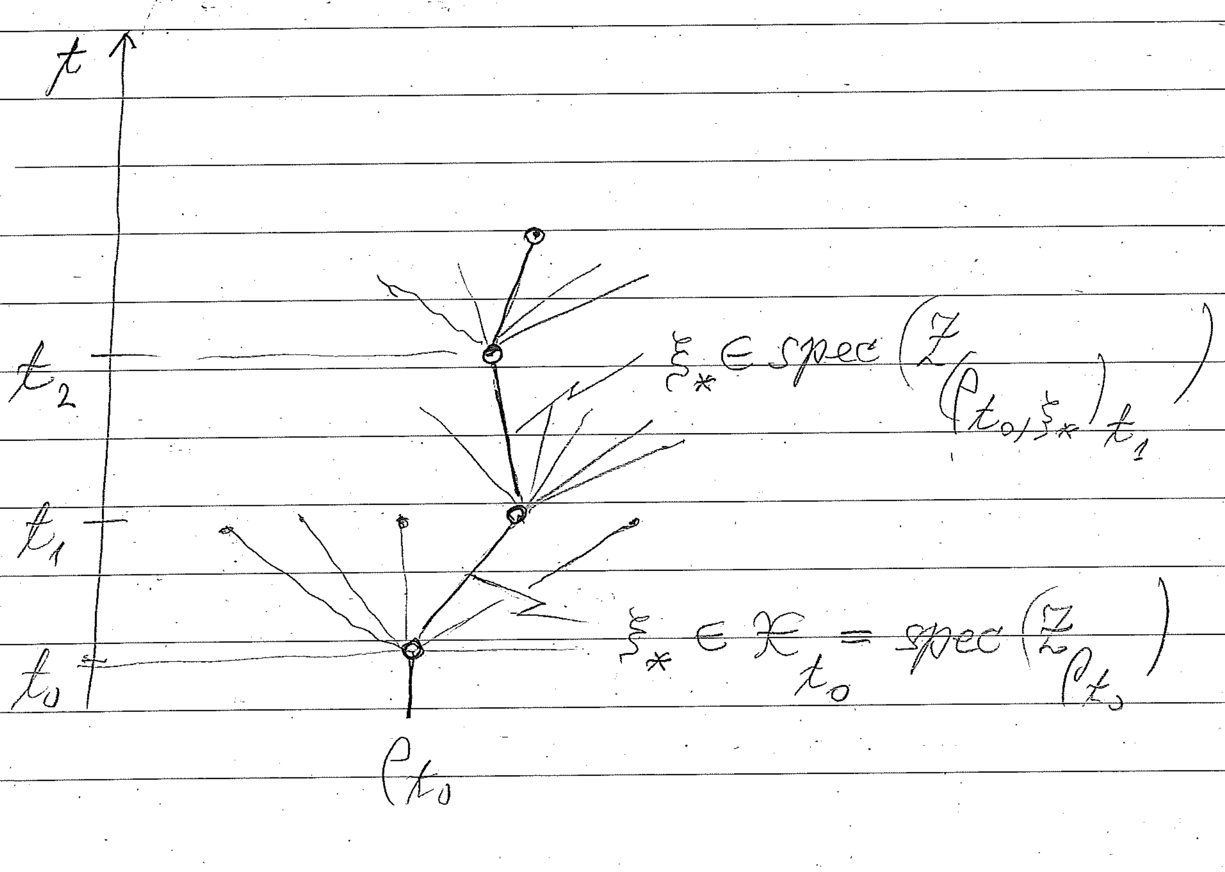
$$\Phi(P_i(t)),\ i = 1,2,3,\cdots,$$

are, in general, neither projections, nor orthogonal; and in general

$$[\Phi(P_i(t)), \Phi(P_j(t))] \neq 0,$$

$i \neq j$. $\Longrightarrow$ (4.34) in type I situation

Consequence: The ETH picture of time evolution of states: A NC stochastic branching process.



What is the mathematics appropriate to describe this?

Special case: S non-autonomous;

Z_{ρ_t} non-trivial only for discrete times.

$\mathcal{E}^{-}_{\geq t}$: type I, $\forall t$

Then (lemma):

$$Z_{\rho_t} = \langle \text{functions of } \rho_t \rangle$$

generated by spectral projections of density matrix ρ_t.

Mixed states generated by entanglement with "lost probes" (probes = degrees of freedom that, after some time, are no longer observable).

→ ETH processes generated by Lindblad ops.; (see also Chap. 5)