5. Indirect Non-Demolition Measurements

(↗ Davies, Kraus, Maassen & Kümmerer)

We start with an example: Experiment by group of Haroche & Raimond (↗ Guerlin et al.)

P : a cavity filled in which a standing wave (= coheren state) of e.m. field is excited

E : "environment/equipment"

consists of: (1) Probes $A_1, A_2, A_3, \ldots$ = identical Rydberg atoms, all prepared in the same superposition of two highly excited internal states, $|\uparrow\rangle$ and $|\downarrow\rangle$; (e.g., initial state of every atom $= |\uparrow\rangle$.)

Atoms A_i, $i = 1,2,3,\ldots$, are independent of each other; do not interact with each other.

During time interval $[(m-1)\tau, m\tau)$, m^{th} atom streams through cavity P and is subsequently subjected to a projective measurement in:

(2) a detector D (τ: duration of a measurement cycle)

Measurement in D corresponds, for example, to measuring value, $\xi = \pm 1$, of observable X,

$$X := \sum_{\xi} \xi \, \pi_{\xi} \,,$$

with

$$\left.\begin{aligned} \xi = 1 &\Leftrightarrow \pi_{1} = |\uparrow\rangle\langle\uparrow| \\ \xi = -1 &\Leftrightarrow \pi_{-1} = |\downarrow\rangle\langle\downarrow| \end{aligned}\right\}$$

$S = P \vee E$

$\mathcal{O}_P = \{ \text{functions of } N \}$,

where N is the photon number operator assoc. with cavity P.

. $\mathcal{O}_E = \langle \mathbb{1}_P \otimes \mathbb{1}_{A_1} \otimes \cdots \otimes X_{A_m} \otimes \mathbb{1}_{A_{m+1}} \otimes \cdots \mid m = 1, 2, 3, \ldots \rangle$

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The Rydberg atoms A_i are out of resonance w.r. to cavity $P \Rightarrow$ they do not emit or absorb a photon from P during their passage through P.

($\Rightarrow$ Time evolution commutes with N!

However, evolution of internal state of every atom; A_m, $m = 1, 2, 3, \ldots$, during its passage through P depends on the number of photons in P.

§5.1

Assume, for simplicity, that only $0, 1, 2, \dots, N < \infty$ photons, (all of the same frequency ω_0), can be excity in P.

Then

$$\mathcal{H}_P = \mathbb{C}^{N+1}$$

$$\mathbb{1}\big/_{\mathcal{H}_P} = \sum_{\nu=0}^{N} \Pi_\nu ,$$

Π_ν ; orth. projection on state with ν photons

$$\hat{N}\big/_{\mathcal{H}_P} = \sum_{\nu=0} \nu \, \Pi_\nu$$

$$\mathcal{H}_{A_m} = \mathbb{C}^2 , \quad \forall\, m$$

$$\mathcal{H}_S = \mathcal{H}_P \otimes \mathcal{H}_D \otimes \mathbb{C}^2_{A_1} \otimes \mathbb{C}^2_{A_2} \otimes \mathbb{C}^2_{A_3} \otimes \cdots ,$$

with reference state

$$\omega = \omega_P \otimes \varphi_D \otimes |\uparrow\rangle \otimes |\uparrow\rangle \otimes |\uparrow\rangle \otimes \cdots , \quad (1)$$

where ω_P is an arbitrary density matrix on $\mathbb{C}^{N+1}$, φ_D is a state of D which it relaxes to exponentially fast after each proj. measurement of observable $X_m = X|_{\mathbb{C}^2_{A_m}}$, $m = 1, 2, 3, \cdots$, with a rate $\gg 1/\tau$;

$|\uparrow\rangle \in \mathbb{C}^2_{A_m}$: initial state of A_m.

Time evolution of S in time-interval $[(m-1)\tau, m\tau)$:

$$U^{(m)} := \sum_{\nu=0}^{N} \Pi_\nu|_{\mathcal{H}_D} \otimes R|_{\mathcal{H}_D} \otimes \mathbb{1}|_{\mathbb{C}^2_{A_1}} \otimes \cdots \otimes \mathbb{1}|_{\mathbb{C}^2_{A_{m-1}}}$$

$$\otimes\, u_\nu|_{\mathbb{C}^2_{A_m}} \otimes \mathbb{1}_{\mathbb{C}^2_{A_{m+1}}} \otimes \cdots,$$

where u_ν is a unitary operator on $\mathbb{C}^2$, R is the time evolution of D driving its state back to φ_D after proj. measurement of $X_m = X|_{\mathbb{C}^2_{A_m}}$.

Born's Rule for probes:

$p(\xi/\nu)$ = probability for X ($= X_m$, for some $m = 1, 2, 3, \cdots$) to have value ξ ($= \pm 1$), given that cavity, P, contains ν photons.

$$\boxed{p(\xi/\nu) = \langle \uparrow / u_\nu^* \, \pi_\xi \, u_\nu / \uparrow \rangle} \qquad (2)$$

$$\Rightarrow \boxed{\sum_\xi p(\xi, \nu) = 1, \forall \nu.}$$

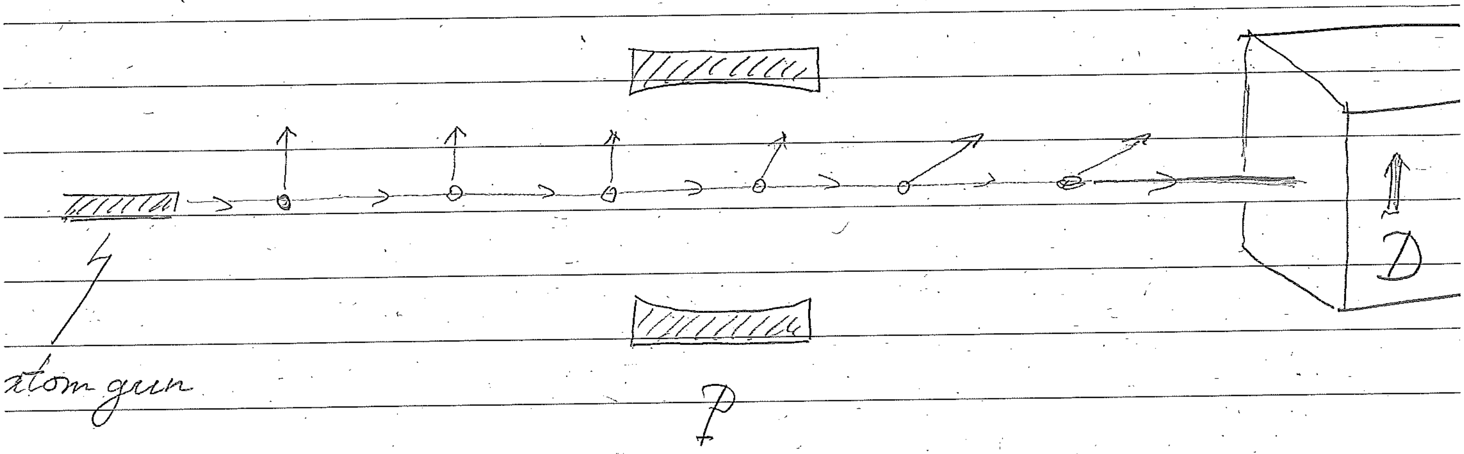
Born's Rule for cavity:

Probability to find ν photons in cavity P, given that P has been prepared in state ω_P?

$$p_\omega(\nu) := \omega_P(\Pi_\nu) \qquad (3)$$

§5.2

~~Sketch of experiment~~



Atom A_m precesses in space spanned by $|\uparrow\rangle |\downarrow\rangle$ during passage through P, $\forall m$. Precession only depends on ν (photon number).

States $|\uparrow\rangle$ of D all occupied $\Rightarrow$ A_m in state $|\uparrow\rangle$ transmitted, A_m in state $|\downarrow\rangle$ absorbed (assuming A_m is a fermion $\rightarrow$ Pauli principle).

"Histories"

History of length r consists of a measurement protocol

$$\underline{\xi}_r = (\underline{\xi}_{r-1}, \xi_r)$$
$$= (\xi_1, \xi_2, \dots, \xi_r)$$

of values ξ_m of observable X_m measured by D on probe A_m after its passage through P.

$\mu(\underline{\xi}_r \mid \nu)$: probability of a history $\underline{\xi}_r$ of length r, given that there are precisely ν photons inside P.

$$\mu(\underline{\xi}_r \mid \nu) = \prod_{m=1}^{r} p(\xi_m \mid \nu) \qquad (4)$$

(see (2), page 7.6).

$\mu_\omega(\underline{\xi}_r)$: probability of history $\underline{\xi}_r$, given that S has been prepared in state ω; (see (1), page 7.4).

Then, by (3), page 7.6, and (4):

$$\mu_\omega(\underline{\xi}_r) = \sum_{\nu=0}^{N} p_\omega(\nu)\, \mu(\underline{\xi}_r \mid \nu) \qquad (5)$$

§ 5.3

Conditional probability for N to have the value ν, given a history $\underline{\xi}_r$ of length r, and given that S has been prepared in state ω:

$$p_\omega^{(r)}(\nu \mid \underline{\xi}) := \frac{p_\omega(\nu)\, \mu(\underline{\xi}_r \mid \nu)}{\mu_\omega(\underline{\xi}_r)} \qquad (6)$$

where $\underline{\xi} := \underline{\xi}_{r=\infty} = (\xi_1, \xi_2, \xi_3, \dots)$

Ξ : space of all histories, $\underline{\xi}$, equipped with σ-algebra of cylinder sets.

<u>Properties of functions $p_\omega^{(r)}(\nu \mid \cdot)$ on Ξ:</u>

(i) $0 \leq p_\omega^{(r)}(\nu \mid \underline{\xi}) \leq 1$,

$$\sum_{\nu=0}^{N} p_\omega^{(r)}(\nu \mid \underline{\xi}) = 1.$$

(ii)

$$p_\omega^{(r)}(\nu \mid \underline{\xi}) = p_\omega(\nu) \frac{\prod_{m=1}^{r} p(\xi_m \mid \nu)}{\sum_{\nu'} p_\omega(\nu') \prod_{m=1}^{r} p(\xi_m \mid \nu')}$$

$$= \underbrace{\frac{p_\omega(\nu) \prod_{m=1}^{r-1} p(\xi_m \mid \nu)}{\sum_{\nu'} p_\omega(\nu') \prod_{m=1}^{r-1} p(\xi_m \mid \nu')}}_{p_\omega^{(r-1)}(\nu \mid \underline{\xi})} \cdot \frac{p(\xi_r \mid \nu)}{\sum_{\nu''} \underbrace{p_\omega(\nu'') \frac{\prod_{m=1}^{r-1} p(\xi_m \mid \nu'')}{\sum_{\nu'} p_\omega(\nu') \prod_{m=1}^{r-1} p(\xi_m \mid \nu')}}_{p_\omega^{(r-1)}(\nu'' \mid \underline{\xi})} \cdot p(\xi_r \mid \nu'')}$$

Thus

$$\boxed{p_\omega^{(r)}(v \mid \underline{\xi}) = p_\omega^{(r-1)}(v \mid \underline{\xi}) \cdot \frac{p(\xi_r \mid v)}{\sum_{v''} p^{(r-1)}(v'' \mid \underline{\xi})\, p(\xi_r \mid v'')}} \tag{7}$$

(iii) Conditional expectation

$$\mathbb{E}_\omega\left[p_\omega^{(r)}(v \mid \cdot) \,\middle|\, \underline{\xi}_{r-1}\right]$$

$$= \left(\sum_{\xi_r} p_\omega^{(r)}(v \mid \underline{\xi})\, \mu_\omega(\underline{\xi}_{r-1}, \xi_r)\right) \Big/ \mu_\omega(\underline{\xi}_{r-1})$$

$$\overset{(4),(5),(6)}{=} \sum_{\xi_r} \frac{p_\omega(v) \prod_{m=1}^{r} p(\xi_m \mid v)}{\cancel{\mu_\omega(\underline{\xi}_r)}} \cdot \frac{\cancel{\mu_\omega(\underline{\xi}_{r-1}, \xi_r)}}{\mu_\omega(\underline{\xi}_{r-1})}$$

$$= p_\omega^{(r-1)}(v \mid \underline{\xi}),$$

because $\sum_{\xi_r} p(\xi_r \mid v) = 1, \ \forall v,$ by (2).

Hence

$$E_\omega\left[p_\omega^{(r)}(\nu|\cdot)\,\middle|\,\underline{\xi}_{r-1}\right] = p_\omega^{(r-1)}(\nu|\underline{\xi}_{r-1}), \quad (8)$$

$$\forall \nu.$$

(i) & (iii) $\Longrightarrow \left\{p_\omega^{(r)}(\nu|\cdot)\right\}_{\nu=0,\dots,N}$ are

<u>bounded martingales</u> on Ξ.

By the <u>martingale convergence theorem</u>,

$$p_\omega^{(r)}(\nu|\cdot) \underset{r\to\infty}{\longrightarrow} p_\omega^{(\infty)}(\nu|\cdot), \quad (9)$$

μ_ω - almost everywhere on Ξ, $\forall \nu$.

Next, using (ii) (see (7), page 7.11), we find

$$p_\omega^{(\infty)}(\nu|\underline{\xi}) = p_\omega^{(\infty)}(\nu|\underline{\xi})\,\frac{p(\xi|\nu)}{\sum_{\nu''} p_\omega^{(\infty)}(\nu''|\underline{\xi},\xi)\,p(\xi|\nu'')}$$

(10)

Suppose $\underline{\xi} \in \Xi$ is such that

$$p_{\omega}^{(\infty)}(\nu \mid \underline{\xi}) \neq 0, \quad \forall \nu \in I_{\underline{\xi}} \subseteq \{0, 1, \dots, N\}$$

Then (i), page 7.11, implies that

$p_{\omega}^{(\infty)}(\cdot \mid \underline{\xi})$ is a probability distribution

on $I_{\underline{\xi}}$. Moreover, (10) then implies that

$$p(\xi \mid \nu) = \sum_{\nu' \in I_{\underline{\xi}}} p_{\omega}^{(\infty)}(\nu' \mid \underline{\xi}, \xi)\, p(\xi \mid \nu')$$

$$= \overline{p(\xi \mid \cdot)}^{\,p_{\omega}^{(\infty)}}, \qquad (11)$$

$\forall \nu \in I_{\underline{\xi}}$.

$*$ Suppose that $p(\xi \mid \nu_1) = p(\xi \mid \nu_2)$ implies that $\nu_1 = \nu_2$. (Remember that ξ only takes 2 values, $\xi = \pm 1$, and $\sum_{\xi = \pm 1} p(\xi \mid \nu) = 1$, $\forall \nu$. Thus, if $*$ holds for $\xi = +1$ it also holds for $\xi = -1$!)

Then (11) implies that

$$I_{\underset{\sim}{\xi}} = \{\nu(\underset{\sim}{\xi})\}, \quad \text{and}$$

$$\boxed{p_\omega^{(\infty)}(\nu \mid \underset{\sim}{\xi}) = \delta_{\nu\, \nu(\underset{\sim}{\xi})},} \qquad (12)$$

for some $\nu(\underset{\sim}{\xi}) \in \{0, 1, \ldots, N\}$.

$\boxed{\text{Purification!}}$

<u>Exercise:</u> Show that, under assumption $*$, (11) implies (12).

[Hint: If $p(\underset{\sim}{\xi} \mid \nu_*) = \max_{\nu \in I_{\underset{\sim}{\xi}}} p(\underset{\sim}{\xi} \mid \nu)$ then

$p(\underset{\sim}{\xi} \mid \nu_*) = p(\underset{\sim}{\xi} \mid \cdot)^{p_\omega^{(\infty)}}$ is compatible with $*$ iff $I_{\underset{\sim}{\xi}} = \{\nu_*\}$.]

<u>Thm.</u> <u>Born's Rule</u>

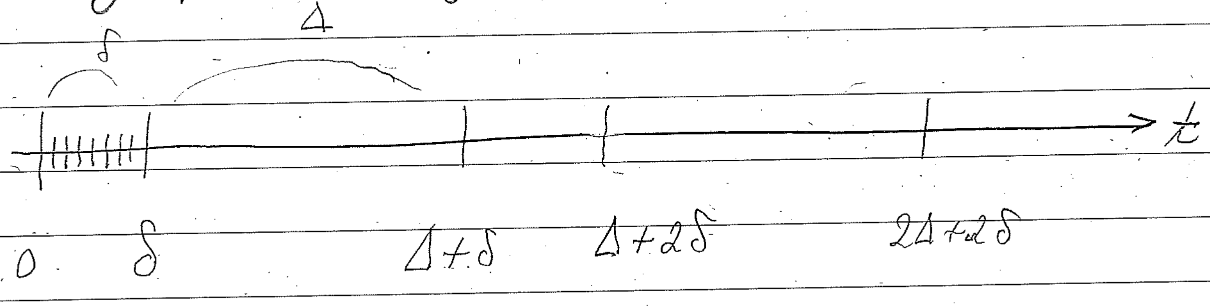
$$\boxed{\mathbb{E}_\omega\left[p_\omega^{(\infty)}(\nu \mid \cdot)\right] = p_\omega(\nu) \equiv \omega_p(\Pi_\nu)}$$

<u>Pf.</u> Exercise

§ 5.4

Quantum jumps & tracks.

Here we consider a limiting regime where $\tau \to 0$; i.e., a very fast rate of shooting probes through cavity P.



$$n = \frac{\delta}{\tau}$$

Limits: $\Delta > 0$ fixed.

(i) For fixed $\delta > 0$, let $\tau \to 0 \Rightarrow$

$$n \to \infty$$

(ii) Then let $\delta \to 0$

Result.

(1) $[H_p, N] \neq 0$;

(2) Proj. measurements of probes lead to indirect measmt. of value of N, (photon number).

Then, in limiting regime #, we obtain a Markov chain, with

- state space $\mathcal{X} = \operatorname{spec} N$
- transition function, Γ, given by

$$\boxed{\Gamma(\nu, \nu') := \left| \exp(-i \Delta H_p)_{\nu, \nu'} \right|^2}$$

$\longrightarrow$ Diffusion process on $\mathcal{X}$.

§ 5.5 General Theory of Non-demolition Measurements.

Recap: "Isolated" system, $S = P \vee E$, char. by: C^*-algebras $\{\mathcal{E}_I\}_{I \subset \mathbb{R}}$ indexed by intervals I of proper time

$$\mathcal{E}_{\geq t} = \overline{\bigvee_{I \subset [t,\infty)} \mathcal{E}_I}^{\|\cdot\|}, \quad \forall t \in \mathbb{R}$$

$$\mathcal{E} = \overline{\bigvee_{I \subset \mathbb{R}} \mathcal{E}_I}^{\|\cdot\|}.$$

States = positive, normalized linear functionals on $\mathcal{E}$.

Choose a "stratum", $\mathcal{S}$, of physically interesting states on $\mathcal{E}$. For $\rho \in \mathcal{S}$, let $(\pi_\rho, \mathcal{H}_\rho, \Omega_\rho)$ be obtained from GNS construction applied to $(\mathcal{E}, \rho)$.

$$\left.\begin{aligned} \mathcal{H}_S &= \bigoplus_{\rho \in \mathcal{S}} \mathcal{H}_\rho \\ \pi_S &= \bigoplus_{\rho \in \mathcal{S}} \pi_\rho \end{aligned}\right\}$$

$\mathcal{E}^-$; weak closure of $\pi_S(\mathcal{E})$

$\mathcal{E}^-_{\geq t}$; " " " $\pi_S(\mathcal{E}_{\geq t})$.

Then

$$\boxed{\mathcal{E}^- \supseteq \overline{\mathcal{E}}_{\geq t} \underset{\neq}{\supset} \overline{\mathcal{E}}_{\geq t'} \supseteq \mathcal{E}_I ,}$$

INFO LOSS!

for any $t' > t$, $I \subseteq [t', \infty)$.

Normal states on $\mathcal{E}^-$ = density matrices on $\mathcal{H}_S$

ω a normal state on $\mathcal{E}^-$

Def.

$$\boxed{\omega_t := \omega \big|_{\mathcal{E}^-_{\geq t}}}$$

Definition. "Observables at future ∞"

$$\mathcal{E}_\infty := \bigcap_{t \in \mathbb{R}} \mathcal{E}^-_{\geq t}$$

Hypothesis 1. "Asymptotic abelianness"

$$\boxed{\mathcal{E}_\infty \subseteq \text{center of } \mathcal{E}^-} \qquad (13)$$

$\mathcal{E}_\infty$: "algebra of facts."

"Instrument" = abelian C^*-algebra, $\mathcal{I}$

with property that, for each time $t \in \mathbb{R}$,

$\exists$ representation, π_t, of $\mathcal{I}$ inside $\mathcal{E}^-_{\geq t}$:

$$\pi_t(\mathcal{I}) \subset \mathcal{E}^-_{\geq t}$$

$$\left(\text{and } \pi_t(\mathcal{I}) \not\subset \mathcal{E}^-_{\geq t'}, \quad t' > t\right)$$

Instruments serve to make observations of events.

"Event algebras."

$$\mathcal{Z}_t = \left\langle \mathcal{Z}_{\rho_t} = \text{center of } \mathcal{C}_{\rho_t} \;\middle|\; \rho \in \mathcal{S} \right\rangle \quad (14)$$

Assume that $\mathcal{S}$ is such that its "event algebras", $\mathcal{Z}_t$, $t \in \mathbb{R}$, are generated by a single instrument, $\mathcal{I}$,

$$\mathcal{I} = \left\langle \pi_\xi \;\middle|\; \xi \in X \right\rangle, \quad (15)$$

with $\pi_\xi = \pi_\xi^* = \pi_\xi^2$, $\forall \xi \in X$,

$$\pi_{\xi_1} \pi_{\xi_2} = \delta_{\xi_1, \xi_2} \pi_{\xi_1}$$

$$\sum_{\xi \in X} \pi_\xi = id_{\mathcal{I}}$$

$$\text{card}(X) < \infty.$$

Namely

$$\boxed{\mathcal{Z}_t \subseteq \pi_t(\mathcal{I}), \quad \forall t} \quad (16)$$

Given an initial state $\omega \in \mathcal{S}$ of S, a history of events is given by

$$\left\{ \Pi_{\xi_i}(t_i) \in \mathcal{Z}^{\omega^{(i-1)}}_{t_i} \subset \mathcal{Z}_{t_i},\ \forall \xi_i \in X \right\}_{i=1,2,3,\dots}$$

where

$$\omega^{(i-1)} := \omega^{(i-2)}_{t_{i-1}}\left(\Pi_{\xi_{i-1}}(t_{i-1})\,(\cdot)\,\Pi_{\xi_{i-1}}(t_{i-1}) \right)$$

$$\times\, \omega^{(i-2)}_{t_{i-1}}\left(\Pi_{\xi_{i-1}}(t_i) \right)$$

$$\omega^{(i-1)}_{t_i} := \omega^{(i-1)}\big|_{\mathcal{E}_{\geq t_i}},$$

$$\omega^{(0)} := \omega$$

$\{t_i\}$ is a sequence of times

$$0 \leq t_1 < t_2 < \cdots < t_n < \cdots$$

$$\xi_i \in X, \quad \forall i$$

Trajectory of states:

$$H_{\underline{\xi}_n} := \prod_{i=1}^{n} \Pi_{\xi_i}(t_i) \qquad (17)$$

State before $n+1^{st}$ event happens:

$$\omega_t^{(n)}(A) := \frac{\omega\left(H_{\underline{\xi}_n} A H^*_{\underline{\xi}_n}\right)}{\omega\left(H_{\underline{\xi}_n} H^*_{\underline{\xi}_n}\right)} \qquad (18)$$

$A \in \mathcal{E}_{\geq t}$, $t_n \leq t \leq t_{n+1}$.

This summarizes the "ETH-approach to QM" in the special case of a single instrument triggering events.

Probability of a history $\underline{\xi}_n$:

$$\mu_\omega(\underline{\xi}_n) := \omega\left(H_{\underline{\xi}_n} H^*_{\underline{\xi}_n}\right) \qquad (19)$$

(LSW-formula)

Note that

$$\sum_{\xi_n \in X} \mu_\omega(\underline{\xi}_n) = \mu_\omega(\underline{\xi}_{n-1})$$

$$\mu_\omega(\phi) = \sum_{\xi \in X} \mu_\omega(\xi) = 1$$

Kolmogorov

$\Longrightarrow$ μ_ω extends to a probability measure on space, Ξ, of ∞ long histories.

(Ξ, Σ) is a measure space, where Σ is the σ-algebra generated by cylinder sets:

$\Delta \in \Sigma$ is a cylinder set iff

$$\Delta = \{\underline{\xi} \in \Xi \mid \xi_i = \eta_i, \text{ for } i \in I\},$$

for some points $\eta_i \in X$, $i \in I$, where I is an arbitrary finite set of indices.

$\Sigma_n \subset \Sigma$ is the σ-algebra generated by all cylinder sets $\Delta \in \Sigma$ of the form

$$\Delta = X^{\times n} \times \tilde{\Delta},$$

$\tilde{\Delta}$ a cylinder set in $\Xi_{\geq n}$,

$$\Xi_{\geq n} := \{\pi_n \underline{\xi} \mid \underline{\xi} \in \Xi\}$$

($$\pi_n \underline{\xi} := (\xi_{n+1}, \xi_{n+2}, \dots).$$

"Tail σ-algebra":

$$\boxed{\Sigma_\infty := \bigcap_{n \in \mathbb{N}} \Sigma_n}$$

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Null-sets and measure classes.

Null-sets are sets $\Delta \in \Sigma$ such that

$$\mu_\omega(\Delta) = 0, \quad \forall \omega \in \mathcal{S}$$

$\mathcal{N}_{\mathcal{S}} :=$ family of all null-sets in Σ.

We say that $\mathcal{N}_\varphi$ determines a "<u>measure class</u>".

$\exists$ state $\Omega_0 \in \mathcal{H}_S$ such that

$(\pi_S, \mathcal{H}_S, \Omega_0)$ results from applying

GNS construction to $(\mathcal{E}, \omega_0)$,

$$\omega_0(A) := \langle \Omega_0, \pi_S(A)\, \Omega_0 \rangle, \quad \forall A \in \mathcal{E}.$$

<u>Lemma.</u>

$\mathcal{N}_\varphi$ consists of <u>all</u> sets $\Delta \in \Sigma$ such that

$$\mu_{\omega_0}(\Delta) = 0$$

Two Σ- measurable functions, f and g, are identified iff

$$f - g = 0, \quad \mu_{\omega_0} \text{- a.e.}$$

$\longrightarrow$ Equivalence relation!