Equivalence classes of bounded functions on Ξ that are measurable w.r.to Σ_∞ form an abelian C^*-algebra denoted by

$$\mathcal{L}_\infty$$

Results:

1. Decomposition of measures μ_ω on spectrum of $\mathcal{L}_\infty$:

$$f \in L^\infty(\Xi)_{N_\varphi} \quad , \quad h \in \mathcal{L}_\infty$$

$(\omega, f) \longmapsto \mathbb{E}_\omega(f \mid \cdot)$: a bounded linear functional on $\mathcal{L}_\infty$, given by

$$\mathbb{E}_\omega(f \mid h) := \int_{\Xi} f(\underline{\xi})\, h(\underline{\xi})\, d\mu_\omega(\underline{\xi}) \qquad (20)$$

If $f \geq 0 \Rightarrow \mathbb{E}_\omega(f \mid \cdot)$ is non-negative, and

$$\| E_\omega (f|\cdot) \| \leq \| f \|_\infty$$

Thus, if f is normalized such that $\int f(\underline{\xi})\, d\mu_\omega(\underset{\sim}{\xi}) = 1$ then we conclude that if $f \geq 0$ $\exists$ probability meas. $\mu_\omega(f|\cdot)$ on spectrum, Ξ_∞, of $\mathcal{L}_\infty$ such that

$$\mu_\omega(f|\Delta) = E_\omega(f|\chi_\Delta),$$

for arb $\Delta \in \Sigma_\infty$.

Next, if $\mu_\omega(f \equiv 1|\Delta) = 0$, for some Δ, then $\mu_\omega(f|\Delta) = 0$, $\forall f \in L^\infty(\Xi)$; (see (20)!)

__Def.__ (i) $dP_\omega := \mu_\omega(1|\cdot)$

(ii)

$$\mu(f|\Delta) := \frac{\mu_\omega(f|\Delta)}{\mu_\omega(1|\Delta)}$$

indep. of ω, up. to mult. by $\chi_{\operatorname{supp} \mu_\omega}$!

Ex.:

For each $\Delta \in \Sigma_\infty$ with $\mu_\omega(1|\Delta) > 0$,

$\mu(f|\Delta)$ is a positive, normalized, linear functional on $L^\infty(\underline{\Xi}) \Rightarrow \exists$ measure

$$d\mu(\underset{\sim}{\xi}|\Delta)$$

such that

$$\mu(f|\Delta) = \int f(\underset{\sim}{\xi})\, d\mu(\underset{\sim}{\xi}|\Delta)$$

Then

$$E_\omega(f|h) = \int_{\underline{\Xi}_\infty} \left(\int_{\underline{\Xi}} f(\underset{\sim}{\xi})\, d\mu(\underset{\sim}{\xi}|\nu) \right) dP_\omega(\nu)$$

extremal measures; are "mutually singular"

Next, we consider a special case:

μ_ω exchangeable

$\longrightarrow$ Apply de Finetti's theorem:

(μ_ω exchangeable)

If
$$\mu_\omega(\xi_1, \dots, \xi_n) = \mu_\omega(\xi_{\pi(1)}, \dots, \xi_{\pi(n)}),$$
$$\forall\, \pi \in \mathcal{S}_n, \ \forall\, (\xi_1, \dots, \xi_n) \in X^{\times n}, \ \forall n$$

then (de Finetti)
$$\mu_\omega(\xi_1, \dots, \xi_n) = \int_{\Xi_\infty} dP_\omega(\nu) \prod_{k=1}^{n} p(\xi_k|\nu),$$

where $p(\xi|\nu)$ is a prob. meas. on X:
$$\left.\begin{aligned} 0 \le p(\xi|\nu) \le 1, \ \forall \xi \in X, \\ \sum_{\xi \in X} p(\xi|\nu) = 1, \end{aligned}\right\}$$

$\forall \nu \in \Xi_\infty$; dP_ω a probability measure on Ξ_∞.

$\longrightarrow$ Back in our <u>example</u>!

$$\underset{\sim}{\xi}^{(m,n)} := (\xi_m, \dots, \xi_n), \quad m < n$$

$$\#_{\underset{\sim}{\xi}^{(m,n)}} := \prod_{i=1}^{m} \prod_{\xi_i} (t_i)$$

More general than exchangeability is

Hypothesis 2. (Ideal Decoherence)

$$\sum_{\xi_i \in X} H_{\underline{\xi}} H_{\underline{\xi}}^{*} = H_{\underline{\xi}(1,i)} H_{\underline{\xi}(i+1,\cdot)} H^{*}_{\underline{\xi}(i+1,\cdot)} H^{*}_{\underline{\xi}(1,i)}$$

as an identity between operators on $\mathcal{H}_S$.

Let $f \in L^{\infty}(\Xi)$ only depend on finitely many ξ_i; e.g., on $(\xi_1, \dots, \xi_n)$, $n < \infty$.

We define a linear map $\Phi : L^{\infty}(\Xi) \to \mathcal{E}^{-}$,

$\Phi : f \longmapsto \Phi(f) \in \mathcal{E}^{-}$, $f \in L^{\infty}(\Xi)$, by

$$\Phi(f) := \sum_{\xi_1 \cdots \xi_n} f(\xi_1, \dots, \xi_n) H_{\underline{\xi}(1,n)} H^{*}_{\underline{\xi}(1,n)}$$

Then

$$\boxed{\omega(\Phi(f)) = \sum_{\underline{\xi}} f(\underline{\xi}) \mu_{\omega}(\underline{\xi})}$$

We have that

$$\|\Phi(f)\| \leq \|f\|_\infty$$

+ continuity in f in $\|\cdot\|_\infty$ norm.

$$\mathcal{O}_\infty := \{\Phi(f) \mid f \in \mathcal{L}_\infty\}$$

Using hypothesis 2, we see that

$$\boxed{\mathcal{O}_\infty \subseteq \mathcal{E}_\infty}$$

<u>Theorem.</u> (BFFS2)

$\mathcal{O}_\infty$ is an abelian algebra,

and

$$\mathcal{L}_\infty \ni f \longmapsto \Phi(f) \in \mathcal{O}_\infty \subseteq \mathcal{E}_\infty$$

is an algebra homomorphism of abelian C^*-algebras.

<u>Pf.</u> $f \in \mathcal{L}_\infty \Rightarrow f = \lim_{n\to\infty} f_n$

pointwise, with f_n Σ_{j_n} - meas., for some sequence $j_n \to \infty$

Let $g \in L^\infty(\Xi)$, dep. on finitely many $\xi_1, \dots, \xi_n$. Then

$$\Phi(f_n g) = \sum_{\underline{\xi}_n,\, \underline{\xi} \geq j_n} f_n(\underline{\xi}) g(\underline{\xi})$$

$$\Pi_{\xi_1} \cdots \Pi_{\xi_n} \Pi_{\xi_{j_n}} \cdots \Pi_{\xi_{j_n}} \Pi_{\xi_n} \cdots \Pi_{\xi_1}$$

Taking the limit $n \to \infty$ on both sides:

$$\Phi(f \cdot g) = \sum_{\underline{\xi}_n} g(\underline{\xi}) \Pi_{\xi_1} \cdots \Pi_{\xi_n} \underset{\mathcal{E}_\infty}{\underset{\cap}{\Phi(f)}} \Pi_{\xi_n} \cdots \Pi_{\xi_1}$$

$$= \Phi(f)\, \Phi(g).$$

$\hookrightarrow$ Completes proof of Thm.

For $f \in \mathcal{L}_\infty$, $f \geq 0$,

$$\left.\begin{array}{l} 0 \leq \Phi(f) \in \mathcal{E}_\infty \\ \Rightarrow 0 \leq \Phi(f)^{1/2} \in \mathcal{E}_\infty \end{array}\right\}$$

Let g depend only on finitely many ξ_i's; and use hypothesis 2:

$$\mu_\omega(f \cdot g) = \sum_{\underline{\xi} \in \Xi} g(\underline{\xi})\, \omega\Big(H_{\underline{\xi}} \underbrace{\Phi(f)}_{\in \mathcal{E}_\infty} H_{\underline{\xi}}^*\Big)$$

$$= \sum_{\underline{\xi} \in \Xi} g(\underline{\xi})\, \omega\big(\Phi(f)^{1/2} H_{\underline{\xi}} H_{\underline{\xi}}^* \Phi(f)^{1/2}\big)$$

$$= \mu_\omega(f) \underbrace{\sum_{\underline{\xi} \in \Xi} g(\underline{\xi})\, \mu_{\omega^f}(\underline{\xi})}_{\equiv \mu_{\omega^f}(g)}$$

~~where~~

$$\omega^f(a) := \frac{\omega\big(\Phi(f)^{1/2} a\, \Phi(f)^{1/2}\big)}{\underbrace{\omega(\Phi(f))}_{= \mu_\omega(f)}}$$

Ergodic Decomposition! (See Thm. 1.6, BFFS2)

Perturbations of Non-Demolition Measurements

(Diffusive Evolution of States)

Isolated system: $S = P \vee E$,

P: subsystem of interest (e.g. cavity in H-R)

E: environment/exp. equipment

Assume that E consists of independent probes, A_i, interacting, one after another, with P and being subject to a measurement afterwards; (A_i e.g. Rydberg atoms; $\nearrow$ H-R).

States of S: Density matrices, ρ_S, acting on $\mathcal{H}_S$, with

$$\mathcal{H}_S = \mathcal{H}_P \otimes \mathcal{H}_E \tag{21}$$

States of P:

$$\rho_P = tr_E \, \rho_S , \tag{22}$$

Hilbert space of a single probe A_i

$$\mathcal{H}_{A_i} \cong \mathbb{C}^k, \quad k < \infty, \quad \forall i.$$

Initial state of every probe is

$$\varphi_0 \in \mathbb{C}^k$$

Reference state in $\mathcal{H}_E$:

$$\bigotimes_{i=1}^{\infty} \varphi_0^{(i)}, \quad i \text{ indexes probes.}$$

$\mathcal{H}_E$ = completion of linear span of vectors

$$\bigotimes_{i=1}^{\infty} \psi^{(i)}, \qquad (23)$$

with $\psi^{(i)} = \varphi_0^{(i)}$, except for finitely many i.

$\exists$ single instrument, $\mathcal{I}$, given by a s.a. op. X on $\mathbb{C}^k$, with

$$X = \sum_{\xi \in \mathcal{X}} \xi \Pi_\xi, \qquad (24)$$

$\mathcal{X}$ = spec X assumed to be simple; i.e., $|\mathcal{X}| = k$; same instrument for all probes.

During every sufficiently small time interval $(s,t) \subset \mathbb{R}$, only (at most) one probe, A_i, interacts with P. At a time t_i, this interaction is interrupted and the observable X is measured for A_i, with result $\xi_i \in \mathcal{X}$.

The initial state of every probe A_i, before it starts to interact with P, is given by φ_0; i.e., by the density matrix

$$P_0^{(i)} = |\varphi_0\rangle\langle\varphi_0|, \qquad (25)$$

$i = 1, 2, 3, \ldots$

Let $P^{(i-1)}_{P,+}$ be the state of P after it has interacted with $A_1, \dots, A_{i-1}$ and right before it starts to interact with A_i. The time-evolution of the state of P coupled to A_i from time t_{i-1} to time t_i, right before X is measured for A_i, is given by

$$P^{(i)}_{P,-} := U(t_i, t_{i-1})\, P^{(i-1)}_{P} \otimes P^{(i)}_{0}\, U(t_i, t_{i-1})^{*} \quad (26)$$

At time t_i, X is measured for A_i, with resulting value $\xi_i \in \mathcal{X}$. <u>After</u> this measurement, the state of P is given by

$$P^{(i)}_{P,+} := \frac{tr_{\mathcal{H}_{A_i}}\left(P^{(i)}_{P,-}\, \mathbb{1} \otimes \Pi_{\xi_i}\right)}{tr_{\mathcal{H}_P \otimes \mathcal{H}_{A_i}}\left(P^{(i)}_{P,-}\, \mathbb{1} \otimes \Pi_{\xi_i}\right)} \quad (27)$$

Assumption on time evolution:

Let N be a s.a. operator on $\mathcal{H}_p$ with (for simplicity) simple spectrum $\{0, 1, \cdots, N\}$; ($\mathcal{H}_p \cong \mathbb{C}^{N+1}$, $N < \infty$)

Let P_ν be the orth. proj. onto the eigenspace of N corresponding to the eigenvalue $\nu \in \{0, 1, \cdots, N\}$.

<u>Hypothesis.</u>

$U(t_i, t_{i-1})$ acts trivially on

$$\bigotimes_{j \neq i} \mathcal{H}_{A_j},$$

while on the factor $\mathcal{H}_p \otimes \mathcal{H}_{A_i}$ it is generated by the Hamiltonian

$$H^{(i)} = H_p \otimes \mathbb{1}_{\mathcal{H}_{A_i}} + \sum_{\nu=0}^{N} P_\nu \otimes h_\nu, \tag{28}$$

with

$$H_P = \varepsilon H, \tag{29}$$

where H is an arbitrary s.a. matrix on $\mathcal{H}_P$; in particular, $[H, N]$ might be non-zero! Furthermore, $0 < \varepsilon \ll 1$.

The matrices h_ν act on $\mathcal{H}_{A_i} \cong \mathbb{C}^k$, are self-adjoint, but otherwise arbitrary.

To simplify our analysis, it is convenient to consider the following simplification:

$$U(t_i, t_{i-1}) := \left(\sum_{\nu=0}^{N} P_\nu \Big|_{\mathcal{H}_P} \otimes U_\nu \Big|_{\mathcal{H}_{A_i}} \right) \times$$

$$\times \exp\left(-i(t_i - t_{i-1}) H_S/\hbar\right)\Big|_{\mathcal{H}_P} \otimes \mathbb{1}\Big|_{\mathcal{H}_{A_i}}, \tag{30}$$

where $\{U_\nu\}_{\nu=0,1,\dots,N}$ are arbitrary, but fixed unitaries on $\mathbb{C}^k$.

Rationale: A_i only briefly interacts with P, during a time interval $[t_{i-1}, t_{i-1}+\delta]$, δ indep. of i, $t_{i-1}+\delta < t_i$, $\forall i$. Its initial state φ_0 is then mapped to $U_\nu \varphi_0$, assuming that P is in an eigenstate of N corresp. to eigenvalue ν.

Let

$$V_\xi := \sum_\nu P_\nu \langle \psi_\xi, U_\nu \varphi_0 \rangle, \qquad (31)$$

where ψ_ξ is the eigenstate of X corresponding to the eigenvalue $\xi \in \mathcal{X}$.

Note that

$$[V_\xi, N] = 0, \ \forall \xi \in \mathcal{X}. \qquad (32)$$

We then find that

$$P^{(i)}_{P,+} = Z^{-1}_{\xi_i} V_{\xi_i} e^{-i(t_i - t_{i-1})H_S/\hbar} P^{(i-1)}_{P,+} \times$$

$$\times e^{i(t_i - t_{i-1})H_S/\hbar} V^*_{\xi_i} \quad (33)$$

where Z_{ξ_i} is chosen such that

$$\mathrm{tr}_{\mathcal{H}_P}\left(P^{(i)}_{P,+}\right) = 1. \quad (34)$$

For concreteness, we now assume that measurement times, $\{t_i\}_{i=1,2,3,\dots}$, are random, with a Poisson distribution of rate $\gamma = 1$:

$$\sum_{\xi\in\mathcal{X}} V^*_{\xi} V_{\xi} = \sum_{\nu=0}^{N} P_{\nu} \sum_{\xi\in\mathcal{X}} \langle U_{\nu}\varphi_0, \psi_{\xi}\rangle\langle\psi_{\xi}, U_{\nu}\varphi_0\rangle$$

$$= \sum_{\nu=0}^{N} P_{\nu} = \mathbb{1}\big|_{\mathcal{H}_P} \quad (35)$$

<u>Def.</u>

$$\langle\nu'|V_{\xi}|\nu\rangle = \delta_{\nu'\nu} V_{\xi}(\nu) := \delta_{\nu'\nu}\langle\psi_{\xi}, U_{\nu}\varphi_0\rangle \quad (36)$$

From now on: $\rho^{(j)} := P^{(j)}_{P,+}$. (5.42)

Between two measurements, at times t_j and t_{j+1}, the state $\rho^{(j)}$ obtained after the j^{th} measurement at time t_j evolves according to Hamiltonian evolution:

$$\rho^{(j)} \longmapsto \rho^{(j)}_t := e^{-i(t-t_j)H_S} \rho^{(j)} e^{i(t-t_j)H_S},$$

$$t_j < t < t_{j+1}$$

Then, at time $t = t_{j+1}$, $\rho^{(j)}_{t_{j+1}}$ changes to

$$\rho^{(j+1)} := \frac{V_{\xi_{j+1}} \rho^{(j)}_{t_{j+1}} V^*_{\xi_{j+1}}}{tr\left(\rho^{(j)}_{t_{j+1}} V^*_{\xi_{j+1}} V_{\xi_{j+1}}\right)} \qquad (37)$$

↪ Yields a trajectory of states:

$$\rho_t(\underline{\xi}_n, \underline{t}_n) = e^{-i(t-t_n)H_S} \rho^{(n)} e^{i(t-t_n)H_S} \qquad (38)$$

<u>Theorem.</u> (BCFFS I)

$$\mathbb{E}\left[\rho_t(\cdot,\cdot)\right] = e^{t\mathcal{L}}\rho,$$

ρ is initial condition, $\mathcal{L}$ is a Lindblad generator given by

$$\boxed{\mathcal{L}\rho = -\frac{i}{\hbar}\,ad_{H_S}(\rho) + \Big(\sum_{\xi\in X} V_\xi \rho V_\xi^*\Big) - \gamma\rho} \quad (39)$$

(38) is called an "<u>unravelling</u>" of the <u>Lindblad evolution</u>.

We are interested in the situation where

$$\boxed{[V_\xi^{\#}, N] = 0,\ \forall \xi,\ \underline{\text{but}}\ [H_S, N] \neq 0} \quad (40)$$

$$H_S := \varepsilon H,$$

ε small

$$\Phi_\xi(a) := V_\xi^* a V_\xi. \quad (41)$$

Super - operator:

$$\tau^s_\varepsilon(\underline{\xi}, t) := e^{i\varepsilon t_1 \mathrm{ad}_H} \Phi_{\xi_1} e^{i\varepsilon(t_2 - t_1)\mathrm{ad}_H} \Phi_{\xi_2} \cdots$$

$$\underset{\substack{\uparrow \\ \text{Poisson process}}}{\Phi_{\xi_{N_s}}} e^{i\varepsilon(s - t_{N_s})\mathrm{ad}_H},$$

$P_{\varepsilon^{-2}t}(\underline{\xi}_n, t_n)$ converges to a jump process on Spec N

Jump process has transition fcts. generated by

$$\frac{2}{\gamma \hbar^2} \mathrm{Re} \frac{|\langle \nu' | H | \nu \rangle|^2}{1 - \sum_{\xi \in X} V_\xi(\nu') \overline{V_\xi(\nu)}} \qquad (\nu \neq \nu')$$

where

$$V_\xi(\nu) = \langle \nu | V_\xi | \nu \rangle$$