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Localization in Geometry and QFTs

Problem Sheet 2: Coadjoint Orbits

Exercise 1 (Coadjoint Orbits).
We consider a compact connected Lie group G with Lie algebra $\mathfrak{g}$ and fix $T \subset G$ a maximal torus with Lie algebra $\mathfrak{t}$. Let $\lambda \in \mathfrak{t}^*$, such that $\mathrm{Stab}(\lambda) = T$, and consider the coadjoint orbit $\mathcal{O}_\lambda$ passing through the point λ.

(a) Compute the G-equivariant cohomology of $\mathcal{O}_\lambda$.
(b) Show that the fixed-point set $\mathcal{O}_\lambda^T$ of the T-action on the coadjoint orbit $\mathcal{O}_\lambda$ is given by the Weyl orbit $W(\lambda)$ passing through the point λ.
(c) From now on we consider the coadjoint orbit $\mathcal{O}_{\lambda+\rho}$, where ρ denotes the Weyl vector. Kirillov showed that there exists a bijection between coadjoint orbits passing through the point $\lambda + \rho$ and representations of G of heighest weight λ. Calculate the Duistermaat-Heckman integral of $\mathcal{O}_{\lambda+\rho}$.
(d) Calculate all of the above for the example $G = \mathrm{SU}(2)$ and $T = \mathrm{U}(1)$.

Exercise 2 (A Physical Interpretation).
The goal of this exercise is to give a physical interpretation of the Duistermaat-Heckmann integration of exercise 1 in the example $G = \mathrm{SU}(2)$ and $T = \mathrm{U}(1)$.

(a) Consider a (quantum) spin particle[1] in a homogeneous magnetic field pointing in the z-direction. Its dynamics is governed by the Hamiltonian $\hat{H} = B\hat{J}_3$, where $\hat{J}_3$ is the operator measuring the z-projection of the particle's spin[2]. Calculate the partition function of a spin-j particle

$$Z(j) = \mathrm{tr}\left(e^{-it\hat{H}}\right).$$

(b) Compute the character of $G = \mathrm{SU}(2)$ using the Weyl character formula

$$\chi_G = \sum_{w \in W} (-1)^{|w|} \frac{e^{i\langle \varepsilon, w(\lambda+\rho)\rangle}}{\prod\limits_{\alpha>0} e^{i\langle\varepsilon,\alpha\rangle/2} - e^{-i\langle\varepsilon,\alpha\rangle/2}},$$

and compare it with the partition function $Z(j)$ for suitable choices.
(c) We now want to consider a classical analogue of the above. For this, we consider a top spinning around the z-axis. As a warm up, one convinces oneself that the configuration space[3] of the system is the two sphere S^2 (the top may spin upside down). Let the dynamics of the classical system be governed by the Hamiltonian $H = \frac{B}{2}(2j + 1)\cos(\theta)$, where θ denotes the polar angle. Calculate the classical partition function

$$Z_{\mathrm{cl}}(j) = \int_{S^2} e^{-itH} d\mu_j$$

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[1] Reminder: quantum spin particles are mathematically described by representations of SU(2) labeled by vectors $|j, m\rangle$, where $j \in \frac{1}{2}\mathbb{Z}$ measures the *spin* of the particle and $m \in \{-j, -j+1, \ldots, j-1, j\} \subset \frac{1}{2}\mathbb{Z}$ measures the degeneracy.

[2] Reminder: $\hat{J}_3$ measures the quantum number m, i.e. one has $\hat{J}_3|j,m\rangle = m|j,m\rangle$

[3] We think of the top as an tilted arrow attached to the origin spinning around the z-axis.

for the measure $d\mu_j = \frac{(2j+1)}{2\pi}\sin(\theta)d\theta d\phi$, and compare your result to the result of the Duistermaat-Heckman integration of exercise 1d.