EQUIVARIANT COHOMOLOGY AND LOCALIZATION WINTER SCHOOL IN LES DIABLERETS JANUARY 2017

VASILY PESTUN

Foundations of equivariant de Rham theory have been laid in two papers by Henri Cartan [1] [2]. The book by Guillemin and Sternberg [3] covers Cartan's papers and treats equivariant de Rham theory from the perspective of supersymmetry. See also the book by Berline-Getzler-Vergne [4], the lectures by Szabo [5] and by Cordes-Moore-Ramgoolam [6], and Vergne's review [7].

See also Kubel-Thom [8, 9] and Kalkman [10] and [11], Meinrenken2003.

1. Lecture: Equivariant cohomology, characteristic classes and equivariant characteristic classes

1.1. Borel model of equivariant cohomology.

Let G be a compact connected Lie group. Let X be a G-manifold, which means that there is a defined action $G \times X \to X$ of the group G on the manifold X.

If G acts freely on X (all stabilizers are trivial) then the quotient X/G is an ordinary manifold on which the usual cohomology theory $H^\bullet(X/G)$ is defined. If the G-action on X is free then G-equivariant cohomology $H^\bullet_G(X)$ are defined as the ordinary cohomology $H^\bullet(X/G)$.

If G-action on X is not free, the naive definition of the equivariant cohomology $H^\bullet_G(X)$ fails because X/G is not an ordinary manifold. If non-trivial stabilizers exist, the corresponding points on X/G are not ordinary points but fractional or stacky points.

A topological definion of the *G-equivariant cohomology* $H_G(X)$ is

$$H^\bullet_G(X) = H^\bullet(X \times_G EG) = H^\bullet((X \times EG)/G)) \tag{1.1}$$

where the space EG, called *universal bundle* [12, 13] is a topological space associated to G with the following properties

(1) The space EG is contractible
(2) The group G acts freely on EG

Because of the property (1) the cohomology theory of X is isomorphic to the cohomology theory of $X \times EG$, and because of the property (2) the group G acts freely on $X \times EG$ and hence the quotient space $(X \times_G EG)$ has a well-defined ordinary cohomology theory.

The space $X \times_G EG$ is called Borel model of homotopy quotient $X//G$, and the definition (1.1) is called Borel equivariant cohomology.

1.2. Equivariant cohomology of a point, classifying space BG and universal bundle EG.

If X is a point pt, the ordinary cohomology theory $H^\bullet(pt)$ is elementary

$$H^n(pt, \mathbb{R}) = \begin{cases} \mathbb{R}, & n = 0 \\ 0, & n > 0 \end{cases} \tag{1.2}$$

but the equivariant cohomology $H^\bullet_G(pt)$ is not trivial. In Borel model we have

$$H^\bullet_G(pt) = H^\bullet(EG/G) = H^\bullet(BG) \tag{1.3}$$

where the quotient space $BG = EG/G$ is called *classifying space*.

The terminology *universal bundle* EG and *classifying space* BG comes from the fact that any smooth principal G-bundle P on a manifold X can be induced by a pullback f^* of the universal principal G-bundle $EG \to BG$ using a suitable smooth map $f : X \to BG$.

$$\begin{array}{ccc} P & \longrightarrow & EG \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BG \end{array}, \qquad P = f^* EG \tag{1.4}$$

1.3. **Chern-Weil theory.** The cohomology groups of the classifying space BG are used to construct *characteristic classes* of principal G-bundles.

Let $\mathfrak{g} = Lie(G)$ be the real Lie algebra of a compact connected Lie group G. Let $\mathbb{R}[\mathfrak{g}]$ be the space of real valued polynomial functions on $\mathfrak{g}$, and let $\mathbb{R}[\mathfrak{g}]^G$ be the subspace of Ad_G invariant polynomials on $\mathfrak{g}$.

For a principal G-bundle over a base manifold X the Chern-Weil morphism

$$\begin{gathered} \mathbb{R}[\mathfrak{g}]^G \to H^\bullet(X, \mathbb{R}) \\ p \mapsto p(F_A) \end{gathered} \tag{1.5}$$

sends an adjoint invariant polynomial p on the Lie algebra $\mathfrak{g}$ to a cohomology class $[p(F_A)]$ in $H^\bullet(X)$ where $F_A = \nabla_A^2$ is the curvature 2-form of a connection ∇_A on the G-bundle. The cohomology class $[p(F_A)]$ does not depend on the choice of the connection A and is called the *characteristic class* of the G-bundle associated to the polynomial $p \in \mathbb{R}[\mathfrak{g}]^G$.

Chern-Weil theory forms a bridge between algebraic topology and differential geometry.

Main theorem. The ring of characteristic classes $\mathbb{R}[\mathfrak{g}]^G$ is isomorphic to the cohomology ring $H^\bullet(BG)$ of the classifying space BG: the Chern-Weil morphism (1.5) is an isomorphism

$$\mathbb{R}[\mathfrak{g}]^G \xrightarrow{\sim} H^\bullet(BG, \mathbb{R}) \tag{1.6}$$

Notice that differential form $p(F_A)$ is an invariant of a principal G-bundle with a connection, i.e. $p(F_A)$ is invariant under the gauge transformations. The cohomology class $[p(F_A)] \in H^*(X, \mathbb{R})$ is a topological invariant of a principal G-bundle, i.e. it is invariant under the gauge transformation and choice of a connection.

See more details for example in Freed-Hopkins work [14] where they prove that the differential forms $p(F_A)$ are the only invariants on the space of principal G-bundles with connections.

1.4. **Example of universal bundle EG and classifying space BG for a circle group S^1.** For the circle group $G = S^1 \simeq U(1)$ the universal bundle ES^1 and classifying space BS^1 can be modelled as

$$\begin{array}{c} ES^1 \\ \downarrow \\ BS^1 \end{array} = \lim_{n \to \infty} \begin{array}{c} S^{2n+1} \\ \downarrow \\ \mathbb{CP}^n \end{array} \tag{1.7}$$

at $n \to \infty$ (possible exercise: prove that S^{2n+1} is contractible as $n \to \infty$).

The expected Chern-Weil isomorphism is

$$\mathbb{C}[\mathfrak{g}]^G \simeq H^\bullet(\mathbb{CP}^\infty, \mathbb{C}) \tag{1.8}$$

We have $\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\epsilon]$, where $\epsilon \in \mathfrak{g}^\vee$ is a linear function on $\mathfrak{g} = Lie(S^1)$ and $\mathbb{C}[\epsilon]$ denotes the free polynomial ring on a generator ϵ. Hence, by Chern-Weil theory we expect $H^\bullet(\mathbb{CP}^\infty, \mathbb{C}) \simeq \mathbb{C}[\tilde{\epsilon}]$ where $\tilde{\epsilon} \in H^2(\mathbb{CP}^\infty, \mathbb{C})$ denotes the image of ϵ under the Chern-Weil homomorphism

$$\tilde{\epsilon} = \epsilon(F_A(\gamma)) \tag{1.9}$$

Explicitly, if we identify group $G = S^1$ with unitary group $U(1)$ of complex numbers z with $z\bar{z} = 1$ and Lie algebra $\mathfrak{g} = Lie(S^1)$ with imaginary numbers such that $z = \exp(\phi)$ for $\phi \in \mathfrak{g}$, then one can check that $(-2\pi\imath)^{-1}\tilde{\epsilon}$ is integral cohomology class, i.e. $(-2\pi\imath)^{-1}\tilde{\epsilon} \in H^2(\mathbb{CP}^\infty, \mathbb{Z})$ (check in the exercise). Conventionally, for an S^1 bundle L, or induced $C^\times$ line bundle L, the first Chern class c_1 is defined as $c_1 = (-2\pi\imath)^{-1}\tilde{\epsilon}$, so that $\int_{\alpha \in H_2(X,\mathbb{Z})} c_1(L) \in \mathbb{Z}$.

In these lectures we prefer not to include the normalization factor $(-2\pi\imath)^{-1}$ into the definition of characteristic classes in $H^2(X, \mathbb{C})$ and define $c_1 = \tilde{\epsilon}$, so that in our conventions for a line bundle $L = \mathcal{O}(d)$ of degree $d \in \mathbb{Z}$ on $X = \mathbb{P}^1$ it holds that

$$d = (-2\pi\imath)^{-1} \int_X F_A(L) \tag{1.10}$$

The cohomological degree of ϵ is 2:

$$\deg \epsilon = \deg F_A(\gamma) = 2 \tag{1.11}$$

Generally, for a compact connected Lie group G we reduce the Chern-Weil theory to the maximal torus $T \subset G$ and identify

$$\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{t}]^{W_G} \tag{1.12}$$

where $\mathfrak{t}$ is the Cartan Lie algebra $\mathfrak{t} = Lie(T)$ and W_G is the Weyl group of G.

To summarize, if G is a connected compact Lie group with Lie algebra $\mathfrak{g} = Lie(G)$, maximal torus T and its Lie algebra $\mathfrak{t} = Lie(T)$, and Weyl group W_G, then it holds

$$\boxed{H^\bullet_G(pt, \mathbb{R}) \simeq H^\bullet(BG, \mathbb{R}) \simeq \mathbb{R}[\mathfrak{g}]^G \simeq \mathbb{R}[\mathfrak{t}]^{W_G}} \tag{1.13}$$

1.5. **Example of classifying space for $U(n)$ and others.** The classifying space for $G = U(n)$ is

$$BU(n) = \lim_{k\to\infty} \mathrm{Gr}_n(\mathbb{C}^{k+n}) \tag{1.14}$$

where $\mathrm{Gr}_n(V)$ denotes the space of n-planes in the vector space V.

exercise: what is the universal bundle EG and classifying space BG for $G = \mathbb{Z}_2$? answer: the univeral bundle is $EG = S^\infty$ which is fibered over the base $\mathbb{RP}^\infty$ with $\mathbb{Z}_2$-fiber.

exercise: what is the universal bundle EG and classifying space BG for $G = SO(n)$? solution: use real grassmanian, see more in [15]

1.6. **Weil model.** Let G be again a compact Lie group. The cohomology $H^\bullet(BG, \mathbb{R})$ of the classifying space BG can also be realized in the Weil algebra

$$\mathcal{W}_\mathfrak{g} := \mathbb{R}[\mathfrak{g}[1] \oplus \mathfrak{g}[2]] = \Lambda\mathfrak{g}^\vee \otimes S\mathfrak{g}^\vee \tag{1.15}$$

Here $\mathfrak{g}[1]$ denotes shift of degree so that elements of $\mathfrak{g}[1]$ are Grassmann. The space of polynomial functions $\mathbb{R}[\mathfrak{g}[1]]$ on $\mathfrak{g}[1]$ is the anti-symmetric algebra $\Lambda\mathfrak{g}^\vee$ of $\mathfrak{g}^\vee$, and the space of polynomial functions $\mathbb{R}[\mathfrak{g}[2]]$ on $\mathfrak{g}[2]$ is the symmetric algebra $S\mathfrak{g}^\vee$ of $\mathfrak{g}^\vee$.

The elements $c \in \mathfrak{g}[1]$ have degree 1 and represent the connection 1-form on the universal bundle. The elements $\phi \in \mathfrak{g}[2]$ have degree 2 and represent the curvature 2-form on the universal bundle. An odd differential on functions on $\mathfrak{g}[1] \oplus \mathfrak{g}[2]$ can be described as an odd vector field δ such that $\delta^2 = 0$. The odd vector field δ of degree 1 represents de Rham differential on the universal bundle

$$\begin{aligned} \delta c &= \phi - \frac{1}{2}[c, c] \\ \delta \phi &= -[c, \phi] \end{aligned} \tag{1.16}$$

which follows from the standard relations between the connection A and the curvature F_A

$$\begin{aligned} dA &= F_A - \frac{1}{2}[A, A] \\ dF_A &= -[A, F_A] \end{aligned} \tag{1.17}$$

This definition implies $\delta^2 = 0$. Indeed,

$$\begin{aligned} \delta^2 c &= \delta\phi - [\delta c, c] = -[c, \phi] - [\phi - \frac{1}{2}[c, c], c] = 0 \\ \delta^2 \phi &= -[\delta c, \phi] + [c, \delta\phi] = -[\phi - \frac{1}{2}[c, c], \phi] - [c, [c, \phi]] = 0 \end{aligned} \tag{1.18}$$

Given a basis T_α on the Lie algebra $\mathfrak{g}$ with structure constants $[T_\beta, T_\gamma] = f^\alpha_{\beta\gamma} T_\alpha$ the differential δ has the form

$$\begin{aligned} \delta c^\alpha &= \phi^\alpha - \frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma \\ \delta \phi^\alpha &= -f^\alpha_{\beta\gamma} c^\beta \phi^\gamma \end{aligned} \tag{1.19}$$

The differential δ can be decomposed into the sum of two differentials

$$\delta = \delta_{\mathrm{K}} + \delta_{\mathrm{BRST}} \tag{1.20}$$

with

$$\begin{aligned} \delta_{\mathrm{K}} \phi &= 0, \qquad \delta_{\mathrm{BRST}} \phi = -[c, \phi] \\ \delta_{\mathrm{K}} c &= \phi, \qquad \delta_{\mathrm{BRST}} c = -\frac{1}{2}[c, c] \end{aligned} \tag{1.21}$$

The differential δ_{BRST} is the BRST differential (Chevalley-Eilenberg differential for Lie algebra cohomology with coefficients in the Lie algebra module $S\mathfrak{g}^\vee$). The differential δ_{K} is the Koszul differential (de Rham differential on $\Omega^\bullet(\Pi\mathfrak{g})$).

The field theory interpretation of the Weil algebra and the differential (1.20) was given in [16] and [17].

The Weil algebra $\mathcal{W}_\mathfrak{g} = \mathbb{R}[\mathfrak{g}[1] \oplus \mathfrak{g}[2]]$ is an extension of the Chevalley-Eilenberg algebra $CE_\mathfrak{g} = \mathbb{R}[\mathfrak{g}[1]] = \Lambda\mathfrak{g}^\vee$ by the algebra $\mathbb{R}[\mathfrak{g}[2]] = S\mathfrak{g}^\vee$ of symmetric polynomials on $\mathfrak{g}$

$$CE_\mathfrak{g} \leftarrow \mathcal{W}_\mathfrak{g} \leftarrow S\mathfrak{g}^\vee \tag{1.22}$$

which is quasi-isomorphic to the algebra of G-equivariant differential forms on the universal bundle

$$G \to EG \to BG \tag{1.23}$$

The duality between the Weil algebra $\mathcal{W}_\mathfrak{g}$ and de Rham algebra $\Omega^\bullet(EG)^G$ of G-equivariant differential forms on EG is provided by the Weil homomorphism

$$\mathcal{W}_\mathfrak{g} \xrightarrow{\nabla_A} \Omega^\bullet(EG)^G \tag{1.24}$$

by a choice of a connection 1-form A (an element $A \in (\Omega^1(EG) \otimes \mathfrak{g})^G$ which satisfies $i_X A = X$ for any $X \in \mathfrak{g}$).

The curvature $F_A \in (\Omega^2(EG) \otimes \mathfrak{g})^G$ of a connection A is $F_A = dA + A \wedge A$. From connection A and field strength F we obtain maps $\mathfrak{g}^\vee \to \Omega^1(EG)$ and $\mathfrak{g}^\vee \to \Omega^2(EG)$

$$\begin{aligned} c^\alpha &\mapsto A^\alpha \\ \phi^\alpha &\mapsto F^\alpha \end{aligned} \tag{1.25}$$

The cohomology of the Weil algebra is trivial

$$H^n(\mathcal{W}_\mathfrak{g}, \delta, \mathbb{R}) = \begin{cases} \mathbb{R}, & n = 0 \\ 0, & n > 0 \end{cases} \tag{1.26}$$

corresponding to the trivial cohomology of $(\Omega^\bullet(EG)^G, d)$ (indeed, cohomology in $(\Omega^\bullet(EG), d)$ are trivial because the space EG is contractible, and cohomology of $(\Omega^\bullet(EG)^G, d)$ are isomorphic to cohomology of $(\Omega^\bullet(EG), d)$ for compact group G by averaging action).

Now we can compute $H^\bullet_G(pt) = H^\bullet(BG)$ in Weil model. We consider $\Omega^\bullet(BG) = \Omega^\bullet(EG/G)$.

For any principal G-bundle $\pi : P \to P/G$ the differential forms on P in the image of the pullback π^* of the space of differential forms on P/G are called *basic*

$$\Omega^\bullet(P)_{\text{basic}} = \pi^* \Omega^\bullet(P/G) \tag{1.27}$$

Let L_α be the Lie derivative in the direction of a vector field α generated by a basis element $T_\alpha \in \mathfrak{g}$, and i_α be the contraction with the vector field generated by T_α.

An element $\omega \in \Omega^\bullet(P)_{\text{basic}}$ can be characterized by two conditions

(1) ω is invariant on P with respect to the G-action: $L_\alpha \omega = 0$
(2) ω is horizontal on P with respect to the G-action: $i_\alpha \omega = 0$

In the Weil model the contraction operation i_α is realized as

$$\begin{aligned} i_\alpha c^\beta &= \delta_\alpha^\beta \\ i_\alpha \phi^\beta &= 0 \end{aligned} \tag{1.28}$$

and the Lie derivative L_α is defined by the usual relation

$$L_\alpha = \delta i_\alpha + i_\alpha \delta \tag{1.29}$$

From the definition of $\Omega^\bullet(P)_{\text{basic}}$ for the case of $P = EG$ we obtain

$$H^\bullet_G(pt) = H^\bullet(BG, \mathbb{R}) = H^\bullet(\Omega^\bullet(EG)_{\text{basic}}, \mathbb{R}) = H^\bullet(\mathcal{W}_\mathfrak{g}, \delta, \mathbb{R})_{\text{basic}} = (S\mathfrak{g}^\vee)^G \tag{1.30}$$

exercise: show by computation that cohomology of the complex $(\mathcal{W}_\mathfrak{g}, \delta, \mathbb{R})_{\text{basic}}$ is isomorphic to $(S\mathfrak{g}^\vee)^G$

.

More generally, G-equivariant cohomology $H_G(X)$ in the Weil model are computed by replacing the complex $\Omega^\bullet(X \times_G EG, d)$ by $((\Omega^\bullet(X) \otimes \mathcal{W}_\mathfrak{g})_{\text{basic}}, d_W = d \otimes 1 + 1 \otimes \delta)$. The subscript $()_{\text{basic}}$ denotes subspace elements which are G-invariant and which are horizontal with respect to the contraction operator $(i_W)_\alpha$ that acts on $\mathcal{W}_\mathfrak{g}$ as $(i_W)_\alpha c^\beta = \delta_\alpha^\beta$ and on $\Omega^\bullet(X)$ as $(i_W)_\alpha \psi^\mu = v_\alpha^\mu$ where v_α^μ is the vector field generated by G-action on X.

It is clear the Weil morphism applied gives a morphism

$$(\Omega^\bullet(X) \otimes \mathcal{W}_\mathfrak{g})_{\text{basic}} \to \Omega(EG \times X)_{\text{basic}} \simeq \Omega(EG \times_G X) \tag{1.31}$$

The proof that above homomorphism is isomorphism

$$(\Omega^\bullet(X) \otimes \mathcal{W}_\mathfrak{g})_{\text{basic}} \xrightarrow{\simeq} \Omega(EG \times_G X) \tag{1.32}$$

can be found in [3] in theorems 2.5.1.

1.7. **Cartan model.** In applications another model for G-equivariant cohomology $H_G(X)$, called Cartan model, is often used. The Cartan model is based on the complex $(\Omega^\bullet(X) \otimes S\mathfrak{g}^\vee)^G$ and it is obtained from the Weil model by the Mathai-Quillen automorphism of associative algebras

$$\Omega^\bullet(X) \otimes \mathcal{W}_\mathfrak{g} \xrightarrow{\exp(i_\theta)} \Omega^\bullet(X) \otimes \mathcal{W}_\mathfrak{g} \tag{1.33}$$

where i_θ is operator $\Omega^1(X) \to \Omega^0(X) \otimes \Lambda^1\mathfrak{g}^\vee$ defined by $i_\theta \psi^\mu = v^\mu_\alpha c^\alpha$ where ψ^μ is a basis in Ω^1, c^α is a basis in $\Lambda^1\mathfrak{g}^\vee$ and v^μ_α is a vector field of G-action on X.

Conjugating by $\exp(i_\theta)$ of the differential d_W in the Weil model and the contraction i_W in the Weil model gives the new operators, called differential d_C in the Cartan model, and contraction i_C in the Cartan model

$$d_C = e^{i_\theta} d_W e^{-i_\theta} \qquad i_C = e^{i_\theta} i_W e^{-i_\theta} \tag{1.34}$$

After the conjugation, the Cartan contraction i_C acts only on the $\mathcal{W}_\mathfrak{g}$ factor in $\Omega(X) \otimes \mathcal{W}_\mathfrak{g}$.

exercise: check this

.

Consequently, the space of $(\Omega(X) \otimes \mathcal{W}_\mathfrak{g})_{\text{basic}}$ defined in the Cartan model with respect to the contraction i_C is simply

$$(\Omega(X) \otimes \mathcal{W}_\mathfrak{g})_{\text{basic}} \simeq (\Omega(X) \otimes S\mathfrak{g}^\vee)^G \tag{1.35}$$

The differential d_C in the Cartan model is

$$d_C = d + \phi^\alpha i_{v_\alpha} \tag{1.36}$$

where v_α is the vector field acting on X.

exercise: check this, see help in [10] if needed

1.8. **More details on Cartan model.** Recall that $(\Omega^\bullet(X) \otimes S\mathfrak{g}^\vee)^G$ denotes the G-invariant subspace in $(\Omega^\bullet(X) \otimes S\mathfrak{g}^\vee)$ under the G-action induced from G-action on X and adjoint G-action on $\mathfrak{g}$.

It is convenient to think about $(\Omega^\bullet(X) \otimes S\mathfrak{g}^\vee)$ as the space

$$\Omega^{\bullet,0}_{C^\infty,\text{poly}}(X \times \mathfrak{g}) \tag{1.37}$$

of smooth differential forms on $X \times \mathfrak{g}$ of degree 0 along $\mathfrak{g}$ and polynomial along $\mathfrak{g}$.

In (T_a) basis on $\mathfrak{g}$, an element $\phi \in \mathfrak{g}$ is represented as $\phi = \phi^\alpha T_\alpha$. Then (ϕ^α) is the dual basis of $\mathfrak{g}^\vee$. Equivalently ϕ^α is a linear coordinate on $\mathfrak{g}$.

The commutative ring $\mathbb{R}[\mathfrak{g}]$ of polynomial functions on the vector space underlying $\mathfrak{g}$ is naturally represented in the coordinates as the ring of polynomials in generators $\{\phi^\alpha\}$

$$\mathbb{R}[\mathfrak{g}] = \mathbb{R}[\phi^1, \ldots, \phi^{\operatorname{rk}\mathfrak{g}}] \tag{1.38}$$

Hence, the space (1.37) can be equivalently presented as

$$\Omega^{\bullet,0}_{C^\infty,\text{poly}}(X \times \mathfrak{g}) = \Omega^\bullet(X) \otimes \mathbb{R}[\mathfrak{g}] \tag{1.39}$$

Given an action of the group G on any manifold M

$$\rho_g : m \mapsto g \cdot m \tag{1.40}$$

the induced action on the space of differential forms $\Omega^\bullet(M)$ comes from the pullback by the map $\rho_{g^{-1}}$

$$\rho_g : \omega \mapsto \rho^*_{g^{-1}}\omega, \qquad \omega \in \Omega^\bullet(M) \tag{1.41}$$

In particular, if $M = \mathfrak{g}$ and $\omega \in \mathfrak{g}^\vee$ is a linear function on $\mathfrak{g}$, then (1.41) is the *co-adjoint action* on $\mathfrak{g}^\vee$.

The invariant subspace $(\Omega^\bullet(X) \otimes \mathbb{R}[\mathfrak{g}])^G$ forms a complex with respect to the *Cartan differential*

$$d_G = d \otimes 1 + i_\alpha \otimes \phi^\alpha \tag{1.42}$$

where $d : \Omega^\bullet(X) \to \Omega^{\bullet+1}(X)$ is the de Rham differential, and $i_\alpha : \Omega^\bullet(X) \to \Omega^{\bullet-1}(X)$ is the operation of contraction of the vector field on X generated by $T_\alpha \in \mathfrak{g}$ with differential forms in $\Omega^\bullet(X)$.

The Cartan model of the G-equivariant cohomology $H_G(X)$ is

$$H_G(X) = H\left((\Omega^\bullet(X) \otimes \mathbb{R}[\mathfrak{g}])^G, d_G\right) \tag{1.43}$$

To check that $d_G^2 = 0$ on $(\Omega^\bullet(X) \otimes \mathbb{R}[\mathfrak{g}])^G$ we compute d_G^2 on $\Omega^\bullet(X) \otimes \mathbb{R}[\mathfrak{g}]$ and find

$$d_G^2 = L_\alpha \otimes \phi^\alpha \tag{1.44}$$

where $L_\alpha : \Omega^\bullet(X) \to \Omega^\bullet(X)$ is the Lie derivative on X

$$L_\alpha = di_\alpha + i_\alpha d \tag{1.45}$$

along vector field generated by T_α.

The infinitesimal action by a Lie algebra generator T_a on an element $\omega \in \Omega^\bullet(X) \otimes R[\mathfrak{g}]$ is

$$T_\alpha \cdot \omega = (L_\alpha \otimes 1 + 1 \otimes L_\alpha) \cdot \omega \tag{1.46}$$

where $L_\alpha \otimes 1$ is the geometrical Lie derivative by the vector field generated by T_α on $\Omega^\bullet(X)$ and $1 \otimes L_a$ is the coadjoint action on $\mathbb{R}[\mathfrak{g}]$

$$L_\alpha = f^\gamma_{\alpha\beta}\phi^\beta \frac{\partial}{\partial \phi^\gamma} \tag{1.47}$$

If ω is a G-invariant element, $\omega \in (\Omega^\bullet(X) \otimes R[\mathfrak{g}])^G$, then

$$(L_\alpha \otimes 1 + 1 \otimes L_\alpha)\omega = 0 \tag{1.48}$$

Therefore, if $\omega \in (\Omega^\bullet(X) \otimes R[\mathfrak{g}])^G$ it holds that

$$d_G^2\omega = (1 \otimes \phi^\alpha L_\alpha)\omega = \phi^\alpha f^\gamma_{\alpha\beta}\phi^\beta \frac{\partial \alpha}{\partial \phi^c} = 0 \tag{1.49}$$

by the antisymmetry of the structure constants $f^\gamma_{\alpha\beta} = -f^\gamma_{\beta\alpha}$. Therefore $d_G^2 = 0$ on $(\Omega^\bullet(X) \otimes \mathbb{R}[\mathfrak{g}])^G$.

The grading on $\Omega^\bullet(X) \otimes \mathbb{R}[\mathfrak{g}]$ is defined by the assignment

$$\deg d = 1 \quad \deg i_{v_\alpha} = -1 \quad \deg \phi^\alpha = 2 \tag{1.50}$$

which implies

$$\deg d_G = 1 \tag{1.51}$$

Let

$$\Omega^n_G(X) = \oplus_k (\Omega^{n-2k} \otimes \mathbb{R}[\mathfrak{g}]^k)^G \tag{1.52}$$

be the subspace in $(\Omega(X) \otimes \mathbb{R}[\mathfrak{g}])^G$ of degree n according to the grading (1.50).

Then

$$\dots \xrightarrow{d_G} \Omega^n_G(X) \xrightarrow{d_G} \Omega^{n+1}_G(X) \xrightarrow{d_G} \dots \tag{1.53}$$

is a differential complex. The equivariant cohomology groups $H^\bullet_G(X)$ in the Cartan model are defined as the cohomology of the complex (1.53)

$$H^\bullet_G(X) \equiv \operatorname{Ker} d_G / \operatorname{Im} d_G \tag{1.54}$$

In particular, if $X = pt$ is a point then

$$H^\bullet_G(pt) = \mathbb{R}[\mathfrak{g}]^G \tag{1.55}$$

in agreement with (1.30).

If x^μ are coordinates on X, and $\psi^\mu = dx^\mu$ are Grassman coordinates on the fibers of ΠTX, we can represent the Cartan differential (1.42) in the notations more common in quantum field theory traditions

$$\begin{aligned} \delta x^\mu &= \psi^\mu \\ \delta \psi^\mu &= \phi^\alpha v^\mu_\alpha \end{aligned} \qquad \delta\phi = 0 \tag{1.56}$$

where v^μ are components of the vector field on X generated by a basis element T_α for the G-action on X. In quantum field theory, the coordinates x^μ are typically coordinates on the infinite-dimensional space of bosonic fields, and ψ^μ are typically coordinates on the infinite-dimensional space of fermionic fields.

Without restriction to $()_{\text{basic}}$ complex in the Cartan model, the Cartan differential (1.34) on $\Omega(X) \otimes \mathcal{W}_{\mathfrak{g}}$ can be presented in coordinate notations similar to (1.56) as follows

$$\begin{aligned} \delta x^\mu &= \psi^\mu + c^\alpha v^\mu_\alpha & \qquad \delta c^\alpha &= \phi^\alpha - \frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma \\ \delta\psi^\mu &= \phi^\alpha v^\mu_\alpha + \partial_\nu v^\mu_\alpha c^\alpha \psi^\nu & \qquad \delta\phi^\alpha &= -f^\alpha_{\beta\gamma} c^\beta \phi^\gamma \end{aligned} \tag{1.57}$$

Restriction to $()_{\text{basic}}$ complex in Cartan model means setting $c^\alpha = 0$, and then the differential (1.57) reduces to the differential (1.56).

1.9. **Equivariant characteristic classes in Cartan model.** Let G and T be compact connected Lie groups.

We consider a T-equivariant G-principal bundle $\pi : P \to X$. This means that an equivariant T-action is defined on P compatible with the G-bundle structure of $\pi : P \to X$. One can take that G acts from the right and T acts from the left.

The compatibility means that T-action on the total space of P

- commutes with the projection map $\pi : P \to X$, so that $\pi(gp) = g\pi(p)$ for $g \in G$ and $p \in P$
- commutes with the G action on the fibers of $\pi : P \to P$, so that $(tp)g = g(pt)$ for $t \in T, g \in G, p \in P$

A T-equivariant characteristic class associates to a T-equivariant principal G-bundle P a cohomology class in $H^\bullet_T(X)$.

A topological definition of T-equivariant characteristic class which is based on homotopy quotients and classifying spaces (like Borel topological model for equivariant cohomologies and Chern-Weil construction of characteristic classes) is the following. Given a principal G-bundle bundle $P \to X$ we consider homotopy quotient $P \times_T ET \to X \times_T ET$ which is again principal G-bundle.

By definition, a T-equivariant characteristic class of principal G-bundle $P \to X$ is a characteristic class of principal G-bundle $P \times_T ET \to X \times_T ET$

Given a connection on G-bundle P we can actually construct equivariant differential form that represents the cohomology class in $H^\bullet_T(X)$ like in the Chern-Weil theory for G-bundles combined with Cartan model for T-equivariant cohomologies.

Let $D_A = d + A$ be a T-invariant connection on a T-equivariant G-bundle P. Here the connection A is a $\mathfrak{g}$-valued 1-form on the total space of P. A T-invariant connection always exists by the averaging procedure for compact Lie group T.

Then we define the T-equivariant connection in the Cartan model for T-equivariant cohomologies

$$D_{A,T} = D_A + \epsilon^a i_{v_a} \tag{1.58}$$

Here where ϵ^a is a basis on the dual Lie algebra of T, and i_{v_a} is the operation of contraction with the vector field generated by T-action on P.

Then we define T-equivariant curvature as follows

$$F_{A,T} = (D_{A,T})^2 - \epsilon^a \otimes \mathcal{L}_{v_a} \tag{1.59}$$

We see that $F_{A,T}$ acts by multiplication on an element from $\Omega^2_T(X) \otimes \mathfrak{g}$, while the terms which corresponds to the differential operators are cancelled.

$$F_{A,T} = F_A - \epsilon^a \otimes \mathcal{L}_{v_a} + [\epsilon^a \otimes i_{v_a}, 1 \otimes D_A] = F_A + \epsilon^a i_{v_a} A \tag{1.60}$$

In fact that term $\epsilon^a i_{v_a} A$, which is an element of $\Omega^0(P) \otimes \mathfrak{g} \otimes \mathfrak{t}^\vee$ can be interpreted as a moment map for T-action on P.

exercise: check that the moment map $\mu = \epsilon^a i_{v_a} A$ is G-equivariant and T-equivariant

We conclude that T-equivariant curvature is a deformation of an ordinary curvature F_A for a principal G-bundle that takes value in $\Omega^2(P) \otimes \mathfrak{g}$ and T-moment map that takes value in $\Omega^0(P) \otimes \mathfrak{g} \otimes \mathfrak{t}^\vee$ with respect to $\mathfrak{g}$-valued symplectic form F_A.

exercise: check that the statement agrees with the conventional definition of a moment map: $i_{v_a} F = D(i_{v_a} A)$

Recall that an ordinary characteristic class for a principal G-bundle on X is $[p(F_A)] \in H^{2d}(X)$ for a G-invariant degree d polynomial $p \in \mathbb{R}[\mathfrak{g}]^G$. Here F_A is the curvature of any connection A on the G-bundle.

In the same way, a T-equivariant characteristic class for a principal G-bundle associated to a G-invariant degree d polynomial $p \in \mathbb{R}[\mathfrak{g}]^G$ is $[p(F_{A,T})] \in H^{2d}_T(X)$.

In T-equivariant case, $F_{A,T}$ is the T-equivariant curvature of any T-equivariant connection A on the G-bundle. The T-equivariant curvature $F_{A,T}$ is a sum of two terms, the $\mathfrak{g}$-valued 'symplectic' term $F_{A,T} \in \Omega^2(P) \otimes \mathfrak{g}$, and T-moment map in $\Omega^0(P) \otimes \mathfrak{g} \otimes \mathfrak{t}^\vee$.

Let X^T be the T-fixed point set on $X = P/G$. If the equivariant curvature $F_{A,T}$ is evaluated on P^T, only the vertical component of i_{v_a} (vertical with respect to the principal G-bundle fibration $P \to X$) contributes to the formula (1.60) and v_a pairs with the vertical component of the connection A on the T-fiber of P given by $g^{-1}dg$. The T-action on G-fibers induces the morphism

$$\rho : \mathfrak{t} \to \mathfrak{g} \tag{1.61}$$

and let $\rho(T_a)$ be the images of T_a basis elements of $\mathfrak{t}$.

Restricted to T-fixed points X^T the T-equivariant characteristic class associated to polynomial $p \in \mathbb{R}[\mathfrak{g}]^G$ is

$$p(F_A + \epsilon^a \rho(T_a)) \tag{1.62}$$

In particularly nice case when X^T is of dimension 0, i.e. a set of discrete points, only the moment map μ, i.e. the term $\epsilon^a \rho(T_a) \in \Omega^0(X) \otimes \mathfrak{g} \otimes \mathfrak{t}^\vee$ contributes, and then T-equivariant characteristic class evaluates to

$$p(\epsilon^a \rho(T_a)) \tag{1.63}$$

Hence, in case of $\dim X^T = 0$, the T-equivariant characteristic class defined an adjoint invariant function on $\mathfrak{g}$ is a function on $\mathfrak{t}$ obtained by pullback by morphism (1.61).

In particular, if V is a representation of G and p is the Chern character of the vector bundle V, then if X^T is a point, the T-equivariant Chern characters is an ordinary character of the space V as a T-module.

exercise: give some examples of F-equivariant principal G-bundles constructed from group quotients. suggestion: Let $G \subset F$ be a subgroup. Then $F \to F/G$ is F-equivariant principal G-bundle. Let $H \subset F$ be another subgroup. Then $F \to F/G$ is also H-equivariant principal G-bundle

1.10. **Chern character for $G = GL(n)$.** Let P be a principal $U(n)$ or $GL(n, \mathbb{C})$ bundle over a manifold X. The Weyl group W_G is the permutation group of n eigenvalues $x_1, \dots x_n$.

The Chern character is an adjoint invariant function

$$\mathrm{ch} : \mathfrak{gl}(n, \mathbb{C}) \to \mathbb{C} \tag{1.64}$$

defined as the trace in the defining representation of $GL(n)$ by complex $n \times n$ matrices of the exponential map

$$\mathrm{ch} : x \mapsto \mathrm{tr}\, e^x \tag{1.65}$$

The exponential map is defined by formal series

$$\mathrm{tr}\, e^x = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{tr}\, x^n \tag{1.66}$$

The eigenvalues of the $\mathfrak{gl}(n, \mathbb{C})$ matrix x are called *Chern roots*. In terms of the Chern roots the Chern character is

$$\mathrm{ch}(x) = \sum_{i=1}^{n} e^{x_i} \tag{1.67}$$

exercise: Let E and F be complex vector bundles. Express Chern character of $E \oplus F$ and $E \otimes F$ in terms of Chern Characters of E, F

.

1.11. **Chern classes for $G = GL(n)$.** If $G = U(n)$ or its complexification $G = GL(n, \mathbb{C})$. The Weyl group W_G is the permutation group of n eigenvalues $x_1, \dots x_n$.

$$H^\bullet(BU(n), \mathbb{C}) = \mathbb{C}[\mathfrak{g}]^{\mathrm{ad}\, G} \simeq \mathbb{C}[x_1, \dots, x_n]^{W_{U(n)}} \simeq \mathbb{C}[c_1, \dots, c_n] \tag{1.68}$$

where $(c_1, \dots, c_n)$ are elementary symmetrical monomials called Chern classes

$$c_k = \sum_{i_1 \le \dots \le i_k} x_{i_1} \dots x_{i_k} \tag{1.69}$$

(Recall that in our conventions it is $(-2\pi\imath)^{-k} c_k$ is an integral cohomology class.)

The Chern class c_k is obtained by expansion of the determinant

$$\det(1 + tF_A) = \sum_{k=0}^{n} t^n c_n(F_A) \tag{1.70}$$

in the defining representation by $n \times n$ complex matrices: here F_A is curvature of a connection A on a principal $GL(n, \mathbb{C})$ bundle on X represented by a two-form valued in complex $n \times n$ matrices. A Chern class c_k is a closed degree $2k$ form on X

exercise: check explicitly that c_k is closed from the definition of the curvature $F_A = dA + A \wedge A$

.

In particular

$$c_1(x) = \operatorname{tr} x, \qquad c_n(x) = \det x \tag{1.71}$$

exercise: let E, F be complex vector bundles. Express Chern classes c_k of the direct sum of bundles $E \oplus F$ in terms of Chern classes of E and F. The same question about Chern classes of the dual bundle $E^\vee$

1.12. **Todd class.** Let P be a principal $GL(n, \mathbb{C})$ bundle over a manifold X. The Todd class of $x \in \mathfrak{gl}(n, \mathbb{C})$ is defined to be

$$\operatorname{td}(x) = \det \frac{x}{1 - e^{-x}} = \prod_{i=1}^{n} \frac{x_i}{1 - e^{-x_i}} \tag{1.72}$$

where det is evaluated in the fundamental representation. The ratio evaluates to a series expansion involving Bernoulli numbers B_k

$$\frac{x}{1 - e^{-x}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} B_k x^k = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \dots \tag{1.73}$$

1.13. **The $\hat{A}$ class.** Let P be a principal $GL(n, \mathbb{C})$ bundle over a manifold X. The $\hat{A}$ class of $x \in GL(n, \mathbb{C})$ is defined as

$$\hat{A} = \det \frac{x}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}} = \prod_{i=1}^{n} \frac{x_i}{e^{x_i/2} - e^{-x_i/2}} \tag{1.74}$$

The $\hat{A}$ class is related to the Todd class by

$$\hat{A}(x) = \det e^{-\frac{x}{2}} \operatorname{td} x \tag{1.75}$$

1.14. **Euler class for $G = SO(2n)$ vector bundles.** Let $G = SO(2n)$ be the special orthogonal group which preserves a Riemannian metric $g \in S^2 V^\vee$ on an oriented real vector space V of $\dim_{\mathbb{R}} V = 2n$.

The Euler characterstic class is defined by the adjoint invariant polynomial

$$\operatorname{Pf} : \mathfrak{so}(2n, \mathbb{R}) \to \mathbb{R} \tag{1.76}$$

of degree n on the Lie algebra $\mathfrak{so}(2n)$ called *Pfaffian* and defined as follows. For an element $x \in \mathfrak{so}(2n)$ let $x' \in V^\vee \otimes V$ denote representation of x on V (fundamental representation), so that x' is an antisymmetric $(2n) \times (2n)$ matrix in some orthonormal basis of V. Let

$g \cdot x' \in \Lambda^2 V^\vee$ be the two-form associated by g to x' , and let $v_g \in \Lambda^{2n} V^\vee$ be the standard volume form on V associated to the metric g, and $v_g^* \in \Lambda^{2n} V$ be the dual of v_g. By definition

$$\mathrm{Pf}(x) = \frac{1}{n!} \langle v_g^*, (g \cdot x')^{\wedge n} \rangle \tag{1.77}$$

For example, for the 2×2-blocks diagonal matrix

$$\mathrm{Pf} \begin{pmatrix} 0 & \epsilon_1 & \dots & \dots & 0 & 0 \\ -\epsilon_1 & 0 & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 0 & \epsilon_n \\ 0 & 0 & \dots & \dots & -\epsilon_n & 0 \end{pmatrix} = \epsilon_1 \dots \epsilon_n \tag{1.78}$$

For an antisymmetric $(2n) \times (2n)$ matrix x', the definition implies that $\mathrm{Pf}(x)$ is a degree n polynomial of matrix elements of x which satisfies

$$\mathrm{Pf}(x)^2 = \det x \tag{1.79}$$

1.15. $U(n)$ **complex bundle vs** $SO(2n)$ **real bundle.** The unitary group $G = U(n)$ can be defined as subgroup of $SO(2n)$ which preserves complex structure $J \in \mathrm{End}(\mathbb{R}^{2n})$, $J^2 = -1$.

Hence any $U(n)$ vector bundle can be considered as $SO(2n)$ vector bundle. Let $x \in \mathfrak{g}$ be an element of Lie algebra of $G = U(n)$.

Denote by $x_{\mathfrak{u}(n)}$ representation of x in $n \times n$ anti-hermitian matrices, and denote by $x_{\mathfrak{so}(2n)}$ representation of x by $2n \times 2n$ real anti-symmetric matrices.

Then the following characteristic classes are isomorphic and integral, that is belong to $H^\bullet(X, \mathbb{Z})$

$$\frac{1}{(-2\pi\imath)^n} c_n(x_{\mathfrak{u}(n)}) = \frac{1}{(2\pi)^n} \mathrm{Pf}(x_{\mathfrak{so}(2n)}) \tag{1.80}$$

exercise: show equivalence

Hint: use that $\imath = \sqrt{-1} \in \mathfrak{u}(1)$ is represented by (2×2) anti-symmetric matrix in $\mathfrak{so}(2)$ acting on $\mathbb{R}^2 \simeq \mathbb{C}$ as $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

1.16. **Integral normalization of the Euler class.** Let P be an $SO(2n)$ principal bundle $P \to X$.

In the standard normalization the Euler class $\mathrm{e}(P)$ is defined in such a way that it takes values in $H^{2n}(X, \mathbb{Z})$ and is given by

$$\mathrm{e}(P) = \frac{1}{(2\pi)^n} [\mathrm{Pf}(F)] \tag{1.81}$$

For example, the Euler characteristic of an oriented real manifold X of real dimension $2n$ is an integer number given by

$$\mathrm{e}(X) = \int_X \mathrm{e}(T_X) = \frac{1}{(2\pi)^n} \int_X \mathrm{Pf}(R) \tag{1.82}$$

where R denotes the curvature form of the tangent bundle T_X.

In quantum field theories the definition (1.77) of the Pfaffian is usually realized in terms of a Gaussian integral over the Grassmann (anticommuting) variables θ which satisfy $\theta_i\theta_j = -\theta_j\theta_i$. The definition (1.77) is presented as

$$\mathrm{Pf}(x) = \int d\theta_{2n} \dots d\theta_1 \exp(-\frac{1}{2}\theta_i x_{ij}\theta_j) \tag{1.83}$$

By definition, the integral $[d\theta_{2n} \dots d\theta_1]$ picks the coefficient of the monomial $\theta_1 \dots \theta_{2n}$ of an element of the the Grassman algebra generated by θ.