3. Lecture: Grothendieck-Riemann-Roch-Hirzebruch-Atiyah-Singer Index theorems

3.1. **Index for a holomorphic vector bundle.** For a holomorphic vector bundle E over a complex variety X of $\dim_{\mathbb{C}} X = n$ the index $\operatorname{ind}(\bar\partial, E)$ is defined as

$$\operatorname{ind}(\bar\partial, E) = \sum_{k=0}^{n} (-1)^k \dim H^k(X, E) \tag{3.1}$$

The localization theorem in K-theory gives the index formula of Grothendieck-Riemann-Roch-Hirzebruch-Atiyah-Singer relating the index to the Todd class

$$\boxed{\operatorname{ind}(\bar\partial, E) = \frac{1}{(-2\pi\sqrt{-1})^n} \int_X \operatorname{td}(T^{1,0}_X) \operatorname{ch}(E)} \tag{3.2}$$

Similarly, the index of Dirac operator $\not{D} : S^+ \otimes E \to S^- \otimes E$ from the positive chiral spinors S^+ to the negative chiral spinors S^-, twisted by a vector bundle E, is defined as

$$\operatorname{ind}(\not{D}, E) = \dim \ker \not{D} - \dim \operatorname{coker} \not{D} \tag{3.3}$$

and is given by the Atiyah-Singer index formula

$$\boxed{\operatorname{ind}(\not{D}, E) = \frac{1}{(-2\pi\sqrt{-1})^n} \int_X \hat{A}(T^{1,0}_X) \operatorname{ch}(E)} \tag{3.4}$$

Notice that on a Kahler manifold the Dirac complex

$$\not{D} : S^+ \to S^-$$

is isomorphic to the Dolbeault complex

$$\cdots \to \Omega^{0,p}(X) \xrightarrow{\bar\partial} \Omega^{0,p+1}(X) \to \dots$$

twisted by the square root of the canonical bundle $K = \Lambda^n (T^{1,0}_X)^\vee$

$$\not{D} = \bar\partial \otimes K^{\frac{1}{2}} \tag{3.5}$$

consistently with the relation (1.75) and the Riemann-Roch-Hirzebruch-Atiyah-Singer index formula

3.2. **Euler characteristic from an index formula.** The vector bundle E in the index formula (3.2) can be promoted to a complex

$$\to E^\bullet \to E^{\bullet+1} \to \tag{3.6}$$

In particular, the $\bar\partial$ index of the complex $E^\bullet = \Lambda^\bullet (T^{1,0})^\vee$ of $(\bullet, 0)$-forms on a Kahler variety X equals the Euler characteristic of X

$$e(X) = \operatorname{ind}(\bar\partial, \Lambda^\bullet (T^{1,0})^\vee) = \sum_{q=0}^{n} \sum_{p=0}^{n} (-1)^{p+q} \dim H^{p,q}(X) \tag{3.7}$$

We find

$$\operatorname{ch} \Lambda^\bullet (T^{1,0})^\vee = \prod_{i=1}^{n} (1 - e^{-x_i}) \tag{3.8}$$

where x_i are Chern roots of the curvature of the n-dimensional complex bundle $T_X^{1,0}$. Hence, the Todd index formula (3.2) gives

$$\mathrm{e}(X) = \frac{1}{(-2\pi\imath)^n} \int c_n(T_X^{1,0}) \tag{3.9}$$

The above agrees with the Euler characteristic (1.82)

3.3. **Equivariant index formula (Dolbeault and Dirac).** Let G be a compact connected Lie group.

Suppose that X is a complex variety and E is a holomorphic G-equivariant vector bundle over X. Then the cohomology groups $H^\bullet(X,E)$ form representation of G. In this case the index of E (3.1) can be refined to *an equivariant index* or *character*

$$\mathrm{ind}_G(\bar\partial, E) = \sum_{k=0}^{n} (-1)^k \operatorname{ch}_G H^k(X,E) \tag{3.10}$$

where $\operatorname{ch}_G H^i(X,E)$ is the character of a representation of G in the vector space $H^i(X,E)$. More concretely, the equivariant index can be thought of as a gadget that attaches to G-equivariant holomorphic bundle E a complex valued adjoint invariant function on the group G

$$\mathrm{ind}_G(\bar\partial, E)(g) = \sum_{k=0}^{n} (-1)^k \operatorname{tr}_{H^k(X,E)} g \tag{3.11}$$

on elements $g \in G$. The sign alternating sum (3.11) is also known as *the supertrace*

$$\mathrm{ind}_G(\bar\partial, E)(g) = \operatorname{str}_{H^\bullet(X,E)} g \tag{3.12}$$

The index formula (3.2) is replaced by the equivariant index formula in which characteristic classes are promoted to G-equivariant characteristic classes in the Cartan model of G-equivariant cohomology with differential $d_G = d + \phi^a i_a$ as in (1.42)

$$\mathrm{ind}(\bar\partial, E)(e^{\phi^a T_a}) = \frac{1}{(-2\pi\imath)^n} \int_X \operatorname{td}_G(T_X) \operatorname{ch}_G(E) = \int_X e_G(T_X) \frac{\operatorname{ch}_G E}{\operatorname{ch}_G \Lambda^\bullet T_X^\vee} \tag{3.13}$$

Here $\phi^a T_a$ is an element of Lie algebra of G and $e^{\phi^a T_a}$ is an element of G, and T_X denotes the holomorphic tangent bundle of the complex manifold X.

If the set X^G of G-fixed points is discrete, then applying the localization formula (2.19) to the equivariant index (3.13) we find the equivariant Lefshetz formula

$$\boxed{\mathrm{ind}(\bar\partial, E)(g) = \sum_{x \in X^G} \frac{\operatorname{tr}_{E_x}(g)}{\det_{T_x^{1,0}X}(1 - g^{-1})}} \tag{3.14}$$

The Euler character is cancelled against the numerator of the Todd character.

3.4. **Example of equivariant index on $\mathbb{CP}^1$.** Let X be $\mathbb{CP}^1$ and let $E = \mathcal{O}(n)$ be a complex line bundle of degree n over $\mathbb{CP}^1$, and let $G = U(1)$ equivariantly act on E as follows. Let z be a local coordinate on $\mathbb{CP}^1$, and let an element $t \in U(1) \subset \mathbb{C}^\times$ send the point with coordinate z to the point with coordinate tz so that

$$\operatorname{ch} T_0^{1,0} X = t \qquad \operatorname{ch} T_\infty^{1,0} X = t^{-1} \tag{3.15}$$

where $T^{1,0}_0 X$ denotes the fiber of the holomorphic tangent bundle at $z=0$ and similarly $T^{1,0}_\infty X$ the fiber at $z=\infty$. Let the action of $U(1)$ on the fiber of E at $z=0$ be trivial. Then the action of $U(1)$ on the fiber of E at $z=\infty$ is found from the gluing relation

$$s_\infty = z^{-n} s_0 \tag{3.16}$$

to be of weight $-n$, so that

$$\operatorname{ch} E|_{z=0} = 1, \qquad \operatorname{ch} E|_{z=\infty} = t^{-n} \tag{3.17}$$

Then

$$\operatorname{ind}(\bar\partial, \mathcal{O}(n), \mathbb{CP}^1)(t) = \frac{1}{1-t^{-1}} + \frac{t^{-n}}{1-t} = \frac{1-t^{-n-1}}{1-t^{-1}} = \begin{cases} \sum_{k=0}^n t^{-k}, & n \ge 0 \\ 0, & n=-1, \\ -t\sum_{k=0}^{-n-2} t^k, & n<-1 \end{cases} \tag{3.18}$$

We can check against the direct computation. Assume $n \ge 0$. The kernel of $\bar\partial$ is spanned by $n+1$ holomorphic sections of $\mathcal{O}(n)$ of the form z^k for $k=0,\dots,n$, the cokernel is empty by Riemann-Roch. The section z^k is acted upon by $t \in T$ with weight t^{-k}. Therefore

$$\operatorname{ind}_T(\bar\partial, \mathcal{O}(n), \mathbb{CP}^1) = \sum_{k=0}^n t^{-k} \tag{3.19}$$

Even more explicitly, for illustration, choose a connection 1-form A with constant curvature $F_A = -\frac{1}{2} i n\omega$, denoted in the patch around $\theta = 0$ (or $z=0$) by $A^{(0)}$ and in the patch around $\theta = \pi$ (or $z=\infty$) by $A^{(\pi)}$

$$A^{(0)} = -\frac{1}{2} i n(1-\cos\theta) d\alpha \qquad A^{(\pi)} = -\frac{1}{2} i n(-1-\cos\theta) d\alpha \tag{3.20}$$

The gauge transformation between the two patches

$$A^{(0)} = A^{(\pi)} - in\, d\alpha \tag{3.21}$$

is consistent with the defining E bundle transformation rule for the sections $s^{(0)}, s^{(\pi)}$ in the patches around $\theta = 0$ and $\theta = \pi$

$$s^{(0)} = z^n s^{(\pi)} \qquad A^{(0)} = A^{(\pi)} + z^n dz^{-n}. \tag{3.22}$$

The equivariant curvature F_T of the connection A in the bundle E is given by

$$F_T = -\frac{1}{2} i n(\omega + \epsilon(1-\cos\theta)) \tag{3.23}$$

as can be verified against the definition (1.60) $F_T = F + \epsilon i_v A$. Notice that to verify the expression for the equivariant curvature (3.23) in the patch near $\theta = \pi$ one needs to take into account contributions from the vertical component $g^{-1}dg$ of the connection A on the total space of the principal $U(1)$ bundle and from the T-action on the fiber at $\theta = \pi$ with weight $-n$.

Then

$$\begin{aligned} \operatorname{ch}(E)|_{\theta=0} &= \exp(F_T)|_{\theta=0} = 1 \\ \operatorname{ch}(E)|_{\theta=\pi} &= \exp(F_T)|_{\theta=\pi} = \exp(-in\epsilon) = t^{-n} \end{aligned} \tag{3.24}$$

for $t = \exp(i\epsilon)$ in agreement with (3.18).

A similar exercise gives the index for the Dirac operator on S^2 twisted by a magnetic field of flux n

$$\operatorname{ind}(\not{D}, \mathcal{O}(n), S^2) = \frac{t^{n/2} - t^{-n/2}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} \tag{3.25}$$

where now we have chosen the lift of the T-action symmetrically to be of weight $n/2$ at $\theta = 0$ and of weight $-n/2$ at $\theta = \pi$. Also notice that up to overall multiplication by a power of t related to the choice of lift of the T-action to the fibers of the bundle E, the relation (3.5) holds

$$\operatorname{ind}(\not{D}, \mathcal{O}(n), S^2) = \operatorname{ind}(\bar{\partial}, \mathcal{O}(n-1), \mathbb{CP}^1) \tag{3.26}$$

because on $\mathbb{CP}^1$ the canonical bundle is $K = \mathcal{O}(-2)$.

3.5. **Example of equivariant index on $\mathbb{CP}^m$.** . Let $X = \mathbb{CP}^m$ be defined by the projective coordinates $(x_0 : x_1 : \cdots : x_m)$ and L_n be the line bundle $L_n = \mathcal{O}(n)$. Let $T = U(1)^{(m+1)}$ act on X by

$$(x_0 : x_1 : \ldots x_m) \mapsto (t_0^{-1} x_0 : t_1^{-1} x_1 : \cdots : t_m^{-1} x_m) \tag{3.27}$$

and by t_k^n on the fiber of the bundle L_n in the patch around the k-th fixed point $x_k = 1, x_{i \neq k} = 0$. We find the index as a sum of contributions from $m+1$ fixed points

$$\operatorname{ind}_T(D) = \sum_{k=0}^{m} \frac{t_k^n}{\prod_{j \neq k}(1 - (t_j/t_k))} \tag{3.28}$$

For $n \geq 0$ the index is a homogeneous polynomial in $\mathbb{C}[t_0, \ldots, t_m]$ of degree n representing the character on the space of holomorphic sections of the $\mathcal{O}(n)$ bundle over $\mathbb{CP}^m$.

$$\operatorname{ind}_T(D) = \begin{cases} s_n(t_0, \ldots, t_m), & n \geq 0 \\ 0, & -m \leq n < 0 \\ (-1)^m t_0^{-1} t_1^{-1} \ldots t_m^{-1} s_{-n-m-1}(t_0^{-1}, \ldots, t_m^{-1}), & n \leq -m-1 \end{cases} \tag{3.29}$$

where $s_n(t_0, \ldots, t_m)$ are complete homogeneous symmetric polynomials. This result can be quickly obtained from the contour integral representation of the sum (3.28)

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{dz}{z} \frac{z^n}{\prod_{j=0}^{m}(1 - t_j/z)} = \sum_{k=0}^{m} \frac{t_k^n}{\prod_{j \neq k}(1 - (t_j/t_k))}, \tag{3.30}$$

If $n \geq -m$ we pick the contour of integration $\mathcal{C}$ to enclose all residues $z = t_j$. The residue at $z = 0$ is zero and the sum of residues is (3.28). On the other hand, the same contour integral is evaluated by the residue at $z = \infty$ which is computed by expanding all fractions in inverse powers of z, and is given by the complete homogeneous polynomial in t_i of degree n.

If $n < -m$ we assume that the contour of integration is a small circle around the $z = 0$ and does not include any of the residues $z = t_j$. Summing the residues outside of the contour, and taking that $z = \infty$ does not contribute, we get (3.28) with the $(-)$ sign . The residue at $z = 0$ contributes by (3.29).

Also notice that the last line of (3.29) relates[1] to the first line by the reflection $t_i \to t_i^{-1}$

$$\frac{t_k^n}{\prod_{j \neq k}(1 - t_j/t_k)} = \frac{(-1)^m (t_k^{-1})^{-n-m-1} (\prod_j t_j^{-1})}{\prod_{j \neq k}(1 - t_j^{-1}/t_k^{-1})} \tag{3.31}$$

[1]Thanks to Bruno Le Floch for the comment

which is the consequence of the Serre duality on $\mathbb{CP}^m$.

3.6. **Equivariant index on general flag manifold and Weyl formula for the character.** The $\mathbb{CP}^1$ in example (3.25) can be thought of as a flag manifold $SU(2)/U(1)$, and (3.18) (3.25) as characters of $SU(2)$-modules. For index theory on general flag manifolds $G_{\mathbb{C}}/B_{\mathbb{C}}$, that is Borel-Weyl-Bott theorem[2], the shift of the form (3.26) is a shift by the Weyl vector $\rho = \sum_{\alpha>0} \alpha$ where α are positive roots of $\mathfrak{g}$.

The index formula with localization to the fixed points on a flag manifold is equivalent to the Weyl character formula.

The generalization of formula (3.25) for generic flag manifold appearing from a co-adjoint orbit in $\mathfrak{g}^*$ is called *Kirillov character formula* [18], [19], [20].

Let G be a compact simple Lie group. The Kirillov character formula equates the T-equivariant index of the Dirac operator $\operatorname{ind}_T(D)$ on the G-coadjoint orbit of the element $\lambda + \rho \in \mathfrak{g}^*$ with the character χ_λ of the G irreducible representation with highest weight λ.

The character χ_λ is a function $\mathfrak{g} \to \mathbb{C}$ determined by the representation of the Lie group G with highest weight λ as

$$\chi_\lambda : X \mapsto \operatorname{tr}_\lambda e^X, \qquad X \in \mathfrak{g} \tag{3.32}$$

Let X_λ be an orbit of the co-adjoint action by G on $\mathfrak{g}^*$. Such orbit is specified by an element $\lambda \in \mathfrak{t}^*/W$ where $\mathfrak{t}$ is the Lie algebra of the maximal torus $T \subset G$ and W is the Weyl group. The co-adjoint orbit X_λ is a homogeneous symplectic G-manifold with the canonical symplectic structure ω defined at point $x \in X \subset \mathfrak{g}^*$ on tangent vectors in $\mathfrak{g}$ by the formula

$$\omega_x(\bullet_1, \bullet_2) = \langle x, [\bullet_1, \bullet_2] \rangle \qquad \bullet_1, \bullet_2 \in \mathfrak{g} \tag{3.33}$$

The converse is also true: any homogeneous symplectic G-manifold is locally isomorphic to a coadjoint orbit of G or central extension of it.

The minimal possible stabilizer of λ is the maximal abelian subgroup $T \subset G$, and the maximal co-adjoint orbit is G/T. Such orbit is called *a full flag manifold*. The real dimension of the full flag manifold is $2n = \dim G - \operatorname{rk} G$, and is equal to the number of roots of $\mathfrak{g}$. If the stabilizer of λ is a larger group H, such that $T \subset H \subset G$, the orbit X_λ is called *a partial flag manifold* G/H. A degenerate flag manifold is a projection from the full flag manifold with fibers isomorphic to H/T.

Flag manifolds are equipped with natural complex and Kahler structure. There is an explicitly holomorphic realization of the flag manifolds as a complex quotient $G_{\mathbb{C}}/P_{\mathbb{C}}$ where $G_{\mathbb{C}}$ is the complexification of the compact group G and $P_{\mathbb{C}} \subset G_{\mathbb{C}}$ is a parabolic subgroup. Let $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ be the standard decomposition of $\mathfrak{g}$ into the Cartan $\mathfrak{h}$ algebra and the upper triangular $\mathfrak{g}_+$ and lower triangular $\mathfrak{g}_-$ subspaces.

The minimal parabolic subgroup is known as Borel subgroup $B_{\mathbb{C}}$, its Lie algebra is conjugate to $\mathfrak{h} \oplus \mathfrak{g}_+$. The Lie algebra of generic parabolic subgroup $P_{\mathbb{C}} \supset B_{\mathbb{C}}$ is conjugate to the direct sum of $\mathfrak{h} \oplus \mathfrak{g}_+$ and a proper subspace of $\mathfrak{g}_-$.

Full flag manifolds with integral symplectic structure are in bijection with irreducible G-representations π_λ of highest weight λ

$$X_{\lambda+\rho} \leftrightarrow \pi_\lambda \tag{3.34}$$

This is known as the Kirillov correspondence in geometric representation theory.

[2]For a short presentation see exposition by J. Lurie at http://www.math.harvard.edu/~lurie/papers/bwb.pdf

Namely, if $\lambda \in \mathfrak{g}^*$ is a weight, the symplectic structure ω is integral and there exists a line bundle $L \to X_\lambda$ with a unitary connection of curvature ω. The line bundle $L \to X_\lambda$ is acted upon by the maximal torus $T \subset G$ and we can study the T-equivariant geometric objects. The Kirillov-Berline-Getzler-Vergne character formula equates the equivariant index of the Dirac operator $\not{D}$ twisted by the line bundle $L \to X_{\lambda+\rho}$ on the co-adjoint orbit $X_{\lambda+\rho}$ with the character χ_λ of the irreducible representation of G with highest weight λ

$$\operatorname{ind}_T(\not{D})(X_{\lambda+\rho}) = \chi_\lambda \tag{3.35}$$

This formula can be easily proven using the Atiyah-Singer equivariant index formula

$$\operatorname{ind}_T(\not{D})(X_{\lambda+\rho}) = \frac{1}{(-2\pi\imath)^n} \int_{X_{\lambda+\rho}} \operatorname{ch}_T(L) \hat{A}_T(T_X) \tag{3.36}$$

and the Atiyah-Bott-Berline-Vergne formula to localize the integral over $X_{\lambda+\rho}$ to the set of fixed points $X^T_{\lambda+\rho}$.

The localization to $X^T_{\lambda+\rho}$ yields the Weyl formula for the character. Indeed, the stabilizer of $\lambda + \rho$, where λ is a dominant weight, is the Cartan torus $T \subset G$. The co-adjoint orbit $X_{\lambda+\rho}$ is the full flag manifold. The T-fixed points are in the intersection $X_{\lambda+\rho} \cap \mathfrak{t}$, and hence, the set of the T-fixed points is the Weyl orbit of $\lambda + \rho$

$$X^T_{\lambda+\rho} = \operatorname{Weyl}(\lambda + \rho) \tag{3.37}$$

At each fixed point $p \in X^T_{\lambda+\rho}$ the tangent space $T_{X_{\lambda+\rho}}|_p$ is generated by the root system of $\mathfrak{g}$. The tangent space is a complex T-module $\oplus_{\alpha>0} \mathbb{C}_\alpha$ with weights α given by the positive roots of $\mathfrak{g}$. Consequently, the denominator of $\hat{A}_T$ gives the Weyl denominator, the numerator of $\hat{A}_T$ cancels with the Euler class $e_T(T_X)$ in the localization formula, and the restriction of $\operatorname{ch}_T(L) = e^\omega$ is $e^{w(\lambda+\rho)}$

$$\operatorname{ind}_T(\not{D})(X_{\lambda+\rho}) = \frac{1}{(-2\pi\imath)^n} \int_{X_{\lambda+\rho}} \operatorname{ch}_T(L) \hat{A}(T_X) = \sum_{w \in W} \frac{e^{iw(\lambda+\rho)\epsilon}}{\prod_{\alpha>0} (e^{\frac{1}{2} i\alpha\epsilon} - e^{-\frac{1}{2} i\alpha\epsilon})} \tag{3.38}$$

We conclude that the localization of the equivariant index of the Dirac operator on $X_{\lambda+\rho}$ twisted by the line bundle L to the set of fixed points $X^T_{\lambda+\rho}$ is precisely the Weyl formula for the character.

3.7. **Equivariant index for differential operators.** See the book by Atiyah [21].

Let E_k be a family of vector bundles over a manifold X indexed by integer $k \in \mathbb{Z}$. Let G be a compact Lie group acting equivariantly $E_k \to X$. The action of G on a bundle E induces canonically a linear action on the space of sections $\Gamma(E)$. For $g \in G$ and a section $\phi \in \Gamma(E)$ the action is

$$(g\phi)(x) = g\phi(g^{-1}x), \quad x \in X \tag{3.39}$$

Let $D_k : \Gamma(E_k) \to \Gamma(E_{k+1})$ be linear differential operators compatible with the G action, and let $\mathcal{E}$ be the complex (that is $D_{k+1} \circ D_k = 0$)

$$\mathcal{E} : \Gamma(E_0) \xrightarrow{D_0} \Gamma(E_1) \xrightarrow{D_1} \Gamma(E_2) \to \dots \tag{3.40}$$

Since D_k are G-equivariant operators, the G-action on $\Gamma(E_k)$ induces the G-action on the cohomology $H^k(\mathcal{E})$. The equivariant index of the complex $\mathcal{E}$ is the virtual character

$$\operatorname{ind}_G(D) : \mathfrak{g} \to \mathbb{C} \tag{3.41}$$

defined by

$$\operatorname{ind}_G(D)(g) = \sum_k (-1)^k \operatorname{tr}_{H^k(\mathcal{E})} g \tag{3.42}$$

If the set X^G of G-fixed points is discrete, the Atiyah-Singer equivariant index formula is

$$\boxed{\operatorname{ind}_G(D) = \sum_{x\in X^G} \frac{\sum_k (-1)^k \operatorname{ch}_G(E_k)|_x}{\det_{T_x X}(1-g^{-1})}} \tag{3.43}$$

For the Dolbeault complex $E_k = \Omega^{0,k}$ and $D_k = \bar\partial : \Omega^{0,k} \to \Omega^{0,k+1}$

$$\to \Omega^{0,\bullet} \xrightarrow{\bar\partial} \Omega^{0,\bullet+1} \to \tag{3.44}$$

the index (3.43) agrees with (3.14) because the numerator in (3.43) decomposes as $\operatorname{ch}_G E \operatorname{ch}_G \Lambda^\bullet T^*_{0,1}$ and the denominator as $\operatorname{ch}_G \Lambda^\bullet T^*_{0,1} \operatorname{ch}_G \Lambda^\bullet T^*_{1,0}$ and the factor $\operatorname{ch}_G \Lambda^\bullet T^*_{0,1}$ cancels out.

For example, the equivariant index of $\bar\partial : \Omega^{0,0}(X) \to \Omega^{0,1}(X)$ on $X = \mathbb{C}_{\langle x\rangle}$ under the $T = U(1)$ action $x \mapsto t^{-1}x$ where $t \in T$ is the fundamental character is contributed by the fixed point $x = 0$ as

$$\operatorname{ind}_T(\mathbb{C}, \bar\partial) = \frac{1-\bar t}{(1-t)(1-\bar t)} = \frac{1}{1-t} = \sum_{k=0}^{\infty} t^k \tag{3.45}$$

where the denominator is the determinant of the operator $1-t$ over the two-dimensional normal bundle to $0 \in \mathbb{C}$ spanned by the vectors ∂_x and $\partial_{\bar x}$ with eigenvalues t and $\bar t$. In the numerator, 1 comes from the equivariant Chern character on the fiber of the trivial line bundle at $x = 0$ and $-\bar t$ comes from the equivariant Chern character on the fiber of the bundle of $(0,1)$ forms $d\bar x$.

We can compare the expansion in power series in t^k of the index with the direct computation. The terms t^k for $k \in \mathbb{Z}_{\geq 0}$ come from the local T-equivariant holomorphic functions x^k which span the kernel of $\bar\partial$ on $\mathbb{C}_{\langle x\rangle}$. The cokernel is empty by the Poincaré lemma. Compare with (3.19).

Similarly, for the $\bar\partial$ complex on $\mathbb{C}^r$ we obtain

$$\operatorname{ind}_T(\mathbb{C}^r, \bar\partial) = \left[\prod_{k=1}^{r} \frac{1}{(1-t_k)}\right]_+ \tag{3.46}$$

where $[]_+$ means expansion in positive powers of t_k.

For application to the localization computation on spheres of even dimension S^{2r} we can compute the index of a certain transversally elliptic operator D which naturally interpolates between the $\bar\partial$-complex in the neighborhood of one fixed point (north pole) of the r-torus T^r action on S^{2r} and the $\bar\partial$-complex in the neighborhood of another fixed point (south pole). The index is a sum of two fixed point contributions

$$\begin{aligned} \operatorname{ind}_T(S^{2r}, D) &= \left[\prod_{k=1}^{r} \frac{1}{(1-t_k)}\right]_+ + \left[\prod_{k=1}^{r} \frac{1}{(1-t_k)}\right]_- \\ &= \left[\prod_{k=1}^{r} \frac{1}{(1-t_k)}\right]_+ + \left[\prod_{k=1}^{r} \frac{(-1)^r t_1^{-1}\dots t_r^{-1}}{(1-t_k^{-1})}\right]_- \end{aligned} \tag{3.47}$$

where $[]_+$ and $[]_-$ denotes the expansions in positive and negative powers of t_k.

3.8. **Atiyah-Singer index formula for a free action G-manifold.** Suppose that a compact Lie group G acts freely on a manifold X and let $Y = X/G$ be the quotient, and let

$$\pi : X \to Y \tag{3.48}$$

be the associated G-principal bundle.

Suppose that D is a $G\times T$ equivariant operator (differential) for a complex $(\mathcal{E}, D)$ of vector bundles E_k over X as in (3.40). The $G \times T$-equivariance means that the complex $\mathcal{E}$ and the operator D are pullbacks by π^* of a T-equivariant complex $\tilde{\mathcal{E}}$ and operator $\tilde{D}$ on the base Y

$$\mathcal{E} = \pi^*\tilde{\mathcal{E}}, \quad D = \pi^*\tilde{D} \tag{3.49}$$

We want to compute the $G \times T$-equivariant index $\mathrm{ind}_{G\times T}(D; X)$ for the complex $(\mathcal{E}, D)$ on the total space X for a $G \times T$ transversally elliptic operator D using T-equivariant index theory on the base Y. We can do that using Fourier theory on G (counting KK modes in G-fibers).

Let R_G be the set of all irreducible representations of G. For each irreducible representation $\alpha \in R_G$ we denote by χ_α the character of this representation, and by W_α the vector bundle over Y associated to the principal G-bundle (3.48). Then, for each irrep $\alpha \in R_G$ we consider a complex $\tilde{\mathcal{E}} \otimes W_\alpha$ on Y obtained by tensoring $\tilde{\mathcal{E}}$ with the vector bundle W_α over Y. The Atiyah-Singer formula is

$$\mathrm{ind}_{G\times T}(D; X) = \sum_{\alpha\in R_G} \mathrm{ind}_T(\tilde{D} \otimes W_\alpha; Y)\chi_\alpha. \tag{3.50}$$

3.9. **Index of Dirac operator on odd-dimensional sphere S^{2r-1}.** We consider an example relevant for localization on odd-dimensional spheres S^{2r-1} which are subject to the equivariant action of the maximal torus T^r of the isometry group $SO(2r)$. The sphere $\pi : S^{2r-1} \to \mathbb{CP}^{r-1}$ is the total space of the S^1 Hopf fibration over the complex projective space $\mathbb{CP}^{r-1}$.

We will apply the equation (3.50) for a transversally elliptic operator D induced from the Dolbeault operator $\tilde{D} = \bar{\partial}$ on $\mathbb{CP}^{r-1}$ by the pullback π^*.

To compute the index of operator $D = \pi^*\bar{\partial}$ on $\pi : S^{2r-1} \to \mathbb{CP}^{r-1}$ we apply (3.50) and use (3.29) and obtain

$$\mathrm{ind}(D, S^{2r-1}) = \sum_{n=-\infty}^{\infty} \mathrm{ind}_T(\bar{\partial}, \mathbb{CP}^{r-1}, \mathcal{O}(n)) = \left[\frac{1}{\prod_{k=1}^r (1-t_k)}\right]_+ + \left[\frac{(-1)^{r-1} t_1^{-1} \dots t_r^{-1}}{\prod_{k=1}^r (1-t_k^{-1})}\right]_- \tag{3.51}$$

where $[]_+$ and $[]_-$ denotes the expansion in positive and negative powers of t_k. See further review in Contribution [22].

3.10. **General Atiyah-Singer index formula.** The Atiyah-Singer index formula for the Dolbeault and Dirac complexes and the equivariant index formula (3.43) can be generalized to a generic situation of an equivariant index of transversally elliptic complex (3.40).

Let X be a real manifold. Let $\pi : T^*X \to X$ be the cotangent bundle. Let $\{E^\bullet\}$ be an indexed set of vector bundles on X and $\pi^*E^\bullet$ be the vector bundles over T^*X defined by the pullback.

The symbol $\sigma(D)$ of a differential operator $D : \Gamma(E) \to \Gamma(F)$ (3.40) is a linear operator $\sigma(D) : \pi^*E \to \pi^*F$ which is defined by taking the highest degree part of the differential operator and replacing all derivatives $\frac{\partial}{\partial x^\mu}$ by the conjugate coordinates p^μ in the fibers of T^*X.

For example, for the Laplacian $\Delta : \Omega^0(X, \mathbb{R}) \to \Omega^0(X, \mathbb{R})$ with highest degree part in some coordinate system $\{x^\mu\}$ given by $\Delta = g^{\mu\nu}\partial_\mu\partial_\nu$ where $g^{\mu\nu}$ is the inverse Riemannian metric, the symbol of Δ is a $\mathrm{Hom}(\mathbb{R}, \mathbb{R})$-valued (i.e. number valued) function on T^*X given by

$$\sigma(\Delta) = g^{\mu\nu}p_\mu p_\nu \tag{3.52}$$

where p_μ are conjugate coordinates (momenta) on the fibers of T^*X.

A differential operator $D : \Gamma(E) \to \Gamma(F)$ is elliptic if its symbol $\sigma(D) : \pi^*E \to \pi^*F$ is an isomorphism of vector bundles π^*E and π^*F on T^*X outside of the zero section $X \subset T^*X$.

The index of a differential operator D depends only on the topological class of its symbol in the topological K-theory of vector bundles on T^*X. The Atiyah-Singer formula for the index of the complex (3.40) is

$$\boxed{\mathrm{ind}_G(D, X) = \frac{1}{(2\pi)^{\dim_{\mathbb{R}} X}} \int_{T^*X} \hat{A}_G(\pi^*T_X)\,\mathrm{ch}_G(\pi^*E^\bullet)} \tag{3.53}$$

Here T^*X denotes the total space of the cotangent bundle of X with canonical orientation such that $dx^1 \wedge dp_1 \wedge dx^2 \wedge dp_2 \dots$ is a positive element of $\Lambda^{\mathrm{top}}(T^*X)$.

Let $n = \dim_{\mathbb{R}} X$. Let π^*T_X denote the vector bundle of dimension n over the total T^*X obtained as pullback of $T_X \to X$ to T^*X. The $\hat{A}_G$-character of π^*T_X is

$$\hat{A}_G(\pi^*T_X) = \det{}_{\pi^*T_X}\left(\frac{R_G}{e^{R_G/2} - e^{-R_G/2}}\right) \tag{3.54}$$

where R_G denotes the G-equivariant curvature of the bundle π^*T_X. Notice that the argument of $\hat{A}$ is $n \times n$ matrix where $n = \dim_{\mathbb{R}} T_X$ (real dimension of X) while if general index formula is specialized to Dirac operator on Kahler manifold X as in (3.4) the argument of the $\hat{A}$-character is an $n \times n$ matrix where $n = \dim_{\mathbb{C}} T_X^{1,0}$ (complex dimension of X).

Even though the integration domain T^*X is non-compact the integral (3.54) is well-defined because of the (G-transversal) ellipticity of the complex π^*E.

For illustration take the complex to be $E_0 \xrightarrow{D} E_1$. Since $\sigma(D) : \pi^*E_0 \to \pi^*E_1$ is an isomorphism outside of the zero section we can pick a smooth connection on π^*E_0 and π^*E_1 such that its curvature on E_0 is equal to the curvature on E_1 away from a compact tubular neighborhood $U_\epsilon X$ of $X \subset T^*X$. Then $\mathrm{ch}_G(\pi^*E^\bullet)$ is explicitly vanishing away from $U_\epsilon X$ and the integration over T^*X reduces to integration over the compact domain $U_\epsilon X$.

It is clear that under localization to the fixed points of the G-action on X the general formula (3.54) reduces to the fixed point formula (3.43). This is due to the fact that the numerator in the $\hat{A}$-character $\det_{\pi^*T_X} R_G = \mathrm{Pf}_{T_{T^*_X}}(R_G)$ is the Euler class of the tangent bundle $T_{T^*_X}$ to T^*X which cancels with the denominator in (2.19), while the restriction of the denominator of (3.54) to fixed points is equal to (3.54) or (3.43), because $\det e^{R_G} = 1$, since R_G is a curvature of the tangent bundle T_X with orthogonal structure group.