## 4. Lecture: Equivariant cohomological field theories

Certain field theories first have been interpreted as cohomological and topological field theories by Witten, see [23], [24].

Often the path integral for supersymmetric field theories can be represented in the form

$$Z = \int_X \alpha \tag{4.1}$$

where $X$ is the superspace (usually of infinite dimension) of all fields of the theory. Moreover, the integrand measure $\alpha$ is closed with respect to an odd operator $\delta$ which is typically constructed as a sum of a supersymmetry algebra generator and a BRST charge

$$\delta\alpha = 0 \tag{4.2}$$

The integrand is typically a product of an exponentiated action functional $S$, perhaps with insertion of a non-exponentiated observable $\mathcal{O}$

$$\alpha = e^{-S}\mathcal{O} \tag{4.3}$$

so that both $S$ and $\mathcal{O}$ are $\delta$-closed

$$\delta S = 0, \qquad \delta\mathcal{O} = 0. \tag{4.4}$$

If $X$ is a supermanifold, such as a total space $\Pi E$ of a vector bundle $E$ (over a base $Y$) with parity inversed fibers, the equivariant Euler characteristic class (Pfaffian) in the Atiyah-Bott-Berline-Vergne formula (2.19) is replaced by the graded (super) version of the Pfaffian. The weights associated to fermionic components contribute inversely compared to the weights associated to bosonic components.

Typically, in quantum field theories the base $Y$ of the bundle $E \to Y$ is the space of fields. Certain differential equations (like BPS equations) are represented by a section $s : Y \to E$. The zero set of the section $s^{-1}(0) \subset Y$ are the field configurations which solve the equations. For example, in topological self-dual Yang-Mills theory (Donaldson-Witten theory) the space $Y$ is the infinite-dimensional affine space of all connections on a principal $G$-bundle on a smooth four-manifold $M_4$. In a given framing, connections are represented by adjoint-valued 1-forms on $M_4$, so $Y \simeq \Omega^1(M_4) \otimes \operatorname{ad}\mathfrak{g}$. A fiber of the vector bundle $E$ at a given connection $A$ on the $G$-bundle on $M_4$ is the space of adjoint-valued two-forms $\Omega^{2+}(M_4) \otimes \operatorname{ad}\mathfrak{g}$. The section $s : \Omega^1(M_4) \otimes \operatorname{ad}\mathfrak{g} \to \Omega_2$ is represented by the self-dual part of the curvature form

$$A \mapsto F_A^+ \tag{4.5}$$

The zeroes of the section $s = 0$ are connections $A$ that are solutions of the equation $F_A^+ = 0$. The integrand $\alpha$ is the Mathai-Quillen representative of the Thom class for the bundle $E \to Y$ like in (2.3) and (2.10). The integral over the space of all fields $X = \Pi E$ localizes to the integral over the zeroes $s^{-1}(0)$ of the section , which in the Donaldson-Witten example is the moduli space of self-dual connections, called *instanton moduli space*.

The functional integral version of the localization formula of Atiyah-Bott-Berline-Vergne has the same formal form

$$\boxed{\int_X \alpha = \int_F \frac{f^*\alpha}{\mathrm{e}(\nu_F)}} \tag{4.6}$$

except that in the quantum field theory version the space $X$ is an infinite-dimensional superspace of fields. The $F$ denotes the localization locus in the space of fields. Let $\Phi_F \subset H^\bullet(X)$

be the Poincaré dual class to $F$, or Thom class of the inclusion $f : F \hookrightarrow X$ which provides the isomorphism

$$f_* : H^\bullet(F) \to H^\bullet(X) \tag{4.7}$$

$$f_* : 1 \mapsto \Phi_F \tag{4.8}$$

Let $\nu_F$ be the normal bundle to $F$ in $X$. In quantum feld theory language the space $F$ is called the moduli space or localization locus, and $\nu_F$ is the space of linearized fluctuations of fields transversal to the localization locus. The cohomology class of $f^*\Phi_F$ in $H^\bullet(F)$ is equal to the Euler class of the normal bundle $\nu_F$

$$[f^*\Phi_F] = e(\nu_F) \tag{4.9}$$

The localization (2.19) from $X$ to $F$ exists whenever the locus $F$ is such that there exists an inverse to the Euler class $e(\nu_F)$ of its normal bundle in $X$. Two examples of such $F$ have been considered above:

(i) if $X = \Pi E$ is the total space of a vector bundle $E \to Y$ with parity inversed fibers, then $F \subset Y \subset X$ can be taken to be the set of zeroes $F = s^-(0)$ of a generic section $s : Y \to E$

(ii) If $X$ is a $G$-manifold for a compact group $G$, then $F$ can be taken to be $F = X^G$, the set of $G$-fixed points on $X$

The formula (4.6) is more general than these examples. In practice, in quantum field theory problems, the localization locus $F$ is found by deforming the form $\alpha$ to

$$\alpha_t = \alpha \exp(-t\delta V) \tag{4.10}$$

Here $t \in \mathbb{R}$ is a deformation parameter, and $V$ is a fermionic functional on the space of fields, such that $\delta V$ has a trivial cohomology class (the cohomology class $\delta V$ is automatically trivial on effectively compact spaces, but on a non-compact space of fields, which usually appears in quantum field theory path integrals, one has to take extra care of the contributions from the boundary at infinity to ensure that $\delta V$ has trivial cohomology class).

If the even part of the functional $\delta V$ is positive definite, then by sending the paratemeter $t \to \infty$ we can see that the integral

$$\int_X \alpha \exp(-t\delta V) \tag{4.11}$$

localizes to the locus $F \subset X$ where $\delta V$ vanishes. Such locus $F$ has an invertible Euler class of its normal bundle in $X$ and the localization formula (4.6) holds.

In some quantum field theory problems, a compact Lie group $G$ acts on $X$ and $\delta$ is isomorphic to an equivariant de Rham differential in the Cartan model of $G$-equivariant cohomology of $X$, so that an element $\mathbf{a}$ of the Lie algeba of $G$ appears as a parameter of the partition function $Z$.

Then the partition function $Z(\mathbf{a})$ can be interpreted as an element of $H^\bullet_{\mathbf{G}}(pt)$, and the Atiyah-Bott-Berline-Vergne localization formula can be applied to compute $Z(\mathbf{a})$.

There are are two types of equivariant partition functions.

In the partition functions of the first type $Z(\mathbf{a})$, the variable $\mathbf{a}$ is a *parameter* of the quantum field theory such as a coupling constant, a background field, a choice of vacuum, an asymptotics of fields or a boundary condition. Such a partition function is typical for a quantum field theory on a non-compact space, such as the Nekrasov partition function of equivariant gauge theory on $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ [25].

In the partition function of the second type, the variable $\mathbf{a}$ is actually a *dynamical field* of the quantum field theory, so that the complete partition function is defined by integration of the partial partition function $\tilde{Z}(\mathbf{a}) \in H_G^\bullet(pt)$

$$Z = \int_{\mathbf{a}\in\mathbf{g}} \mu(\mathbf{a})\tilde{Z}(\mathbf{a}) \tag{4.12}$$

where $\mu(a)$ is a certain adjoint invariant volume form on the Lie algebra $\mathfrak{g}$. The partition function $Z$ of second type is typical for quantum field theories on compact space-times reviewed in [26], such as the partition function of a supersymmetric gauge theory on $S^4$ [27] reviewed in Contribution [28], or on spheres of other dimensions, see summary of results in Contribution [22].

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