Exercises for the course "Quantum curves", I

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Exercise 1. A *ribbon graph* is the graph corresponding to a cell decomposition of a compact oriented surface. It is of type (g,n) if the number of faces is n and the genus of the compact oriented surface is g.

Let

$$R_{g,n} = \bigsqcup_{\substack{\Gamma \text{ ribbon graph} \\ \text{of type}(g,n)}} \mathbb{R}_+^{e(\Gamma)}/\mathrm{Aut}\Gamma;$$

construct the topological space $R_{1,1}$.

Exercise 2. We will show that $\mathcal{M}_{1,1} = \mathbb{H}/SL_2(\mathbb{Z})$

1. Let ω_1, $\omega_2 \in \mathbb{C}^*$ such that $\Im(\omega_1/\omega_2) > 0$. What does this condition mean?

2. We call *periods* two complex numbers ω_1, $\omega_2 \in \mathbb{C}^*$ as in the previous point. A *Weierstrass function* for the periods ω_1, ω_2 is a meromorphic function p over $\mathbb{C}$ such that

   - $p(z + m\omega_1 + n\omega_2) = p(z)$ for integer m and n;
   - $p(z) \sim \frac{1}{z^2}$ for $z \to 0$;
   - the only poles of p are on the points of the lattice generated by ω_1 and ω_2, and such poles are double.

   Show that a Weierstrass function exists and it is unique.

3. Show that

$$p(z) = \frac{1}{z^2} + \sum_{h=2}^{\infty}(2h-1)G_{2h}(\omega_1,\omega_2)z^{2h-2}$$

   for complex numbers $G_{2h}(\omega_1,\omega_2)$.

4. Define $g_2 = 60G_4$ and $g_3 = 140G_6$, show that

$$(p'(z))^2 = 4p(z)^3 - g_2 p(z) - g_3$$

5. We call an *affine elliptic curve* a smooth subvariety of $\mathbb{C}^2$ defined by an equation of the form
$$y^2 = 4x^3 - g_2 x - g_3.$$
Show that the above equation defines an affine elliptic curve if and only if $g_2^3 - 27g_3^2 \neq 0$.

6. For $\tau \in \mathbb{C}$ with $\Im(\tau) > 0$, let
$$E_\tau = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}.$$
Show that E_τ is isomorphic to the projectivization of an affine elliptic curve. We call it simply an *elliptic curve.*

7. Show that $E_\tau \simeq E_{\tau'}$ if and only if $\tau' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau$ for some matrix of $SL(2,\mathbb{Z})$. Conclude that $\mathcal{M}_{1,1} \simeq \mathbb{H}/SL_2(\mathbb{Z})$.

8. Draw a picture of $\mathcal{M}_{1,1}$. Show that it is an orbifold whose group is $\mathbb{Z}/2\mathbb{Z}$ at all points except two, in which it is $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ respectively.