Exercises

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Exercise 1

Consider the Liouville equation of motion

$$\partial_+\partial_-\phi = -\mu e^{2\phi}, \tag{0.1}$$

satisfied by a function $\phi(x_+, x_-)$, with $x_\pm = t \pm x$ which is periodic in x, $\phi(x_+ + 2\pi, x_- - 2\pi) = \phi(x_+, x_-)$. Prove that the general solution can be represented in the form

$$\phi(x_+, x_-) = \frac{1}{2}\log\left(\frac{1}{\mu}\frac{\partial_+ A_+(x_+)\partial_- A_-(x_-)}{(A_+(x_+) - A_-(x_-))^2}\right), \tag{0.2}$$

with $A_\pm(x_\pm)$ smooth, monotonic and quasi-periodic,

$$A_\pm(x_\pm + 2\pi) = e^{\pm 4\pi p} A_\pm(x_\pm). \tag{0.3}$$

To this aim first show that (0.1) implies that there exist two functions $t_{\pm\pm}(x_\pm)$ related to ϕ as

$$t_{\pm\pm}(x_\pm) = (\partial_\pm\phi)^2 - \partial_\pm^2\phi. \tag{0.4}$$

Then show that one can reconstruct a solution ϕ of (0.1) from any given pair of smooth periodic functions $t_{\pm\pm}(x_\pm)$. The first step is to construct two linearly independent solutions $f_i^\pm(x_\pm)$, $i = 1, 2$ of

$$\partial_\pm^2 f_i^\pm(x_\pm) = t_{\pm\pm}(x_\pm) f_i^\pm(x_\pm), \tag{0.5}$$

satisfying in addition

$$f_1^\pm \partial_\pm f_2^\pm - f_2^\pm \partial_\pm f_1^\pm = 1, \qquad \begin{aligned} f_1^\pm(x_\pm + 2\pi) &= e^{\mp 2\pi p} f_1^\pm(x_\pm), \\ f_2^\pm(x_\pm + 2\pi) &= e^{\pm 2\pi p} f_2^\pm(x_\pm). \end{aligned} \tag{0.6}$$

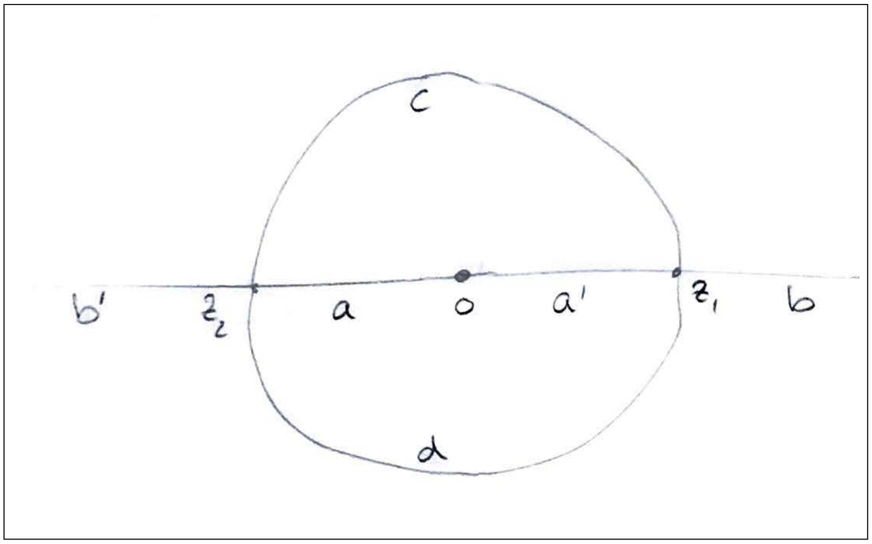
Then construct ϕ via

$$e^{-\phi(x_+, x_-)} = \sqrt{\mu}\,(f_1^+(x_+) f_2^-(x_-) - f_2^+(x_+) f_1^-(x_-)). \tag{0.7}$$

Check that the functions ϕ defined in this way satisfy (0.1). From $f_i^\pm$ construct $A_\pm(x_\pm) = f_2^\pm(x_\pm)/f_1^\pm(x_\pm)$. Show that $A_\pm$ have the required properties. Express $f_i^\pm$ in terms of $A_\pm$, and recover (0.2).

Exercise 2:

Consider the following triangulation of $C_{0,4} := \mathbb{P}^1 \setminus \{0, z_1, z_2, \infty\} \simeq \mathbb{C}^1 \setminus \{0, z_1, z_2\}$:



Find a fundamental domain for the uniformization of the four-punctured sphere such that the edge labelled by letter a is represented by a straight line orthogonal to the real axis in the upper half plane $\mathbb{H}$ contained in the interior of the fundamental domain.

(Recall that the boundary of a fundamental domain can be represented by geodesic arcs, and that geodesics in hyperbolic geometry can be represented by half-circles in $\mathbb{H}$ centered on the real axis $\mathbb{R}$, or by straight lines orthogonal to $\mathbb{R}$).

Extension: Assume (without having to prove it) that the function $t(z) = (\partial_z \phi)^2 - \partial_z^2 \phi$ defined from the metric $ds^2 = e^{2\phi(z,\bar{z})} dz d\bar{z}$ of constant negative curvature on $C_{0,4}$ with punctures at 0, z_1, z_2 and ∞ has the form

$$t(z) = \frac{1}{4z^2} + \frac{1}{4(z-z_1)^2} + \frac{1}{4(z-z_1)^2} - \frac{1}{2z(z-z_2)} + \frac{H}{z(z-z_1)(z-z_2)}. \tag{0.8}$$

We have seen that the uniformising map $A(z)$ can be obtained from two linearly independent solutions of $(\partial_z^2 + t(z)) f_i(z) = 0$, $i = 1, 2$, by setting $A(z) = f_1(z)/f_2(z)$. Assuming $z_i \in \mathbb{R}$ show explicitly that the solutions $f_i(z)$, $i = 1, 2$ can be chosen in such a way that the function $A(z)$ maps to a fundamental domain of the form determined in the first part of the exercise.

Hints: Under the conditions above one may find a pair $f_i(z)$, $i = 1, 2$ of *real* solutions in each interval on the real axis. This means that $A(z) := \mathrm{i} f_1(z)/f_2(z)$ will map this interval to the upper imaginary half-axis. The key to the proof is then to understand the analytic continuation of the function $A(z)$ around 0, z_1 or z_2. To this aim show that for each puncture z_i, $i = 0, 1, 2$,

$z_0 = 0$, there exists a basis for the solutions of the form

$$f_1^{(i)}(z) = \sum_{k=0}^{\infty} f_{1,k}^{(i)}(z - z_i)^k, \tag{0.9}$$

$$f_2^{(i)}(z) = \log(z - z_i) f_1^{(i)}(z) + \sum_{k=0}^{\infty} f_{2,k}^{(i)}(z - z_i)^k. \tag{0.10}$$

Using such a basis one can understand, for example, how the function $A(z)$ defined above in $(z_2, 0)$ can be analytically continued to the interval $(0, z_1)$. To get the full picture note that any two geodesics in $\mathbb{H}$ can be mapped to each other by Möbius transformations.

Exercise 3

Given self-adjoint operators p, q satisfying $[\mathsf{q}, \mathsf{p}] = \frac{\mathrm{i}}{2}$, construct

$$\mathsf{U} = e^{b(2\mathsf{q}-\mathsf{q})}, \qquad \mathsf{U} = e^{b(2\mathsf{q}+\mathsf{q})}.$$

Show at least formally that $\mathsf{UV} = q^2 \mathsf{VU}$ with $q = e^{\pi \mathrm{i} b^2}$.

The function $s_b(x)$ introduced in the lecture satisfies

$$\begin{aligned}
&a) \;\; s_b(x - \mathrm{i}b/2) = 2\cosh(2\pi b x) s_b(x + \mathrm{i}b/2), \\
&b) \;\; |s_b(x)| = 1 \;\text{ for }\; x \in \mathbb{R}, \\
&c) \;\; s_b(x) \;\text{ is analytic for all } x \in \mathbb{C}, \mathrm{Im}(x) < \tfrac{1}{2}(b + b^{-1}).
\end{aligned}$$

Use a)-c) to derive the identity

$$(\mathsf{U} + \mathsf{V})^{is} = e^{2\mathrm{i}sb\mathsf{q}} \frac{s_b(\mathsf{p} + sb)}{s_b(\mathsf{p})},$$

which played an important role in the lecture.

Exercise 4

Use the algorithm formulated in the lecture to compute the length function L_{01} in terms of shear coordinates. This algorithm associates matrices

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{to a left turn,}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \qquad \text{to a right turn,}$$

$$\begin{pmatrix} 0 & e^{f_e/2} \\ -e^{-f_e/2} & 0 \end{pmatrix} \qquad \text{to a path along edge } e.$$

Express the result in terms of $X = e^{f_a/2}$, $Y = e^{(f_d - f_c)/2}$ and the central elements c_k of the Poisson-algebra generated by the shear coordinates, given by the sums of the shear coordinates of all the edges incident into puncture z_k, for $k = 0, 1, 2, \infty$, respectively.

Exercise 5

Quantise the length function L_{01} by first introducing a non-commutative algebra with generators $\mathsf{f}_a, \mathsf{f}_d$, commutation relation $[\mathsf{f}_a, \mathsf{f}_d] = 4\pi \mathrm{i} b^2$, and central elements c_k associated to the punctures z_k, for $k = 0, 1, 2, \infty$, respectively.

One may then define the quantum counterpart L_{01} by replacing in the expression for L_{01} from Exercise 3 monomials of the form $e^{\frac{k}{2} f_a + \frac{l}{2} f_d}$, $k, l \in \mathbb{Z}$, by generators $e^{\frac{k}{2}\mathsf{f}_a + \frac{l}{2}\mathsf{f}_d}$.

Using the function $s_b(x)$ introduced in the lecture which satisfies

$$s_b(x - \mathrm{i}b/2) = 2\cosh(2\pi b x) s_b(x + \mathrm{i}b/2), \tag{0.11}$$

find the eigenfunctions of L_{01}.