

Lecture 1. First order classical field theories

Note Title

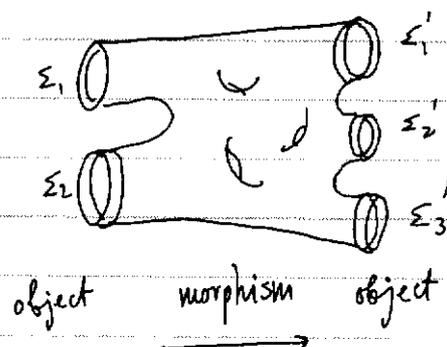
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• The space-time category

- Minkowsky space times. Pseudo Riemannian with the signature $(-, +, \dots, +)$. The boundary $\partial N = \partial_+ N \cup \partial_- N$, $\partial_{\pm} N = (n-1)$ dimensional Riemann
- Riemannian space times. Compact, oriented, n -dim Riemannian manifolds. $\partial N = (n-1)$ -dim, compact oriented (more precisely n -dimensional collar of ∂N :
not , but  $\leftarrow \partial N \times (-\epsilon, \epsilon)$)
- Combinatorial space times. Cell complexes, possibly with metric or other structures.
- Topological space times. Homeomorphism classes of smooth, oriented, compact manifolds (preserving ∂N).

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Convenient to consider as a category



(with collars in the Riemannian category)

- Space of fields F_N (usually forms with values in a vector bundle, sections of a fiber bundle, connections ...). May form a super-manifold if there are classical Fermi fields.

Locally: $X^i(x)$ - forms on the space time N .

- Local 1-st order action

$$S_N[X] = \int_N \omega_i(x) \wedge dX^i + \int_N \beta(x)$$

\uparrow local expression in X ,
 mixed form on N (locally)

- EL_N = solutions to the Euler-Lagrange equations:

$$- \omega_{ij}(x) \wedge dX^j + \partial_i \beta(x) = 0, \quad \omega = d\alpha$$

- The projection to boundary fields

$$\pi: F_N \rightarrow F_{\partial N} \quad (\text{pull-back to the boundary})$$

- Boundary symplectic form

$$\omega_{\partial N} = \int_{\partial N} \omega_{ij}(Y) \wedge \delta Y^i \wedge \delta Y^j$$

\uparrow as forms on ∂N \uparrow as forms on ∂N
 and as differentials of fields

Regular theory: (a) ω is nondegenerate

* $L_N = \pi(EL_N) \subset F_{\partial N}$, is always isotropic

Regular theory: L_N is Lagrangian

* $C_{\partial N} = \pi(\text{solutions to Euler-Lagrange in vicinity of } \partial N) \subset F_{\partial N}$

$C_{\partial N}$ is always co-isotropic

Symmetry

• Collection of vector fields on F_N generating an involutive distribution on EL_N (gauge vector fields).

• The action S_N is invariant with respect to these vector fields:

$$L_v S = 0 \quad \left(\begin{array}{l} L_v = \sum v^i(x) \partial_i \text{ the Lie} \\ \text{derivative along } v \end{array} \right)$$

• Gauge vector fields generate an involutive distribution on $C_{\partial N}$ and on $L_N \subset C_{\partial N}$

• (Space of leaves of this distribution on $C_{\partial N}$) = the symplectic reduction of $C_{\partial N}$.

(Space of leaves through L_N) = image of L_N under the symplectic reduction of $C_{\partial N}$.

Examples:

1) First order classical Lagrangian mechanics

M = configuration space,

• $\mathcal{L}(v, x)$ = the Lagrangian, function on TM

$v \in T_x M$ the velocity

• $\mathcal{L}(v, x)$ = linear in v , $x \in M, v \in T_x M$

$$\mathcal{L}(v, x) = (\alpha(x), v) - \beta(x), \quad \alpha \in \Omega^1(M)$$

• Space time $I = [t_2, t_1]$, flat Riemannian, $F_I = \overset{\infty}{C}(I \rightarrow M)$

↑ fields

• The action functional:

$$\begin{aligned} S[\gamma] &= \int_{t_1}^{t_2} (\alpha(\gamma(t)), \dot{\gamma}(t)) - \int_{t_1}^{t_2} \beta(\gamma(t)) dt = \\ &= \int_{\gamma} \alpha - \int_{t_1}^{t_2} \beta(\gamma(t)) dt \end{aligned}$$

Hamilton-Jacobi action when $M = T^*N$ and $\alpha = \sum_i p_i dq^i$

• Euler-Lagrange equations

$$\omega_{ij}(\gamma(t)) \dot{\gamma}^j(t) - \frac{\partial \beta}{\partial x^i}(\gamma(t)) = 0, \quad \omega = d\alpha = \omega_{ij} dx^i \wedge dx^j$$

Hamiltonian flows on M with $\omega = d\alpha$ generated by the Hamiltonian β .

• Boundary fields = endpoints of $\{\gamma(t)\} = M \times M = F \partial I$
 (t_2, γ_2) , (t_1, γ_1)

Variation of $S \rightarrow \alpha_{\text{tot}} = -\alpha_1 + \alpha_2 \in \Omega^1(M \times M)$

$$\begin{array}{ccc} & M \times M & \\ \swarrow (s,t) & \leftarrow \pi_1 \quad \pi_2 \searrow (s,t) & \\ s & M & t \end{array} \quad \alpha_1 = \pi_1^*(\alpha), \quad \alpha_2 = \pi_2^*(\alpha)$$

Symplectic structure: $\omega = d\alpha_{\text{tot}}$

- $EL_I \subset F_I$, $EL_I = \left\{ \begin{array}{l} \text{Hamiltonian paths generated} \\ \text{by } \beta \end{array} \right\}$
- $L_I \subset M \times M$, $L_I = \left\{ (q_1, q_2) \in M \times M \mid \begin{array}{l} \text{connected by} \\ \text{paths in } EL_I \end{array} \right\}$

Remark. Natural boundary conditions: $L_1 \subset M$, $L_2 \subset M$.

Lagrangian; $\Rightarrow L_1 \times L_2 \subset M \times M$ is Lagrangian

$(L_1 \times L_2) \cap L_I = \{\text{points}\}$ correspond to classical trajectories $\gamma(t)$, s.t. $\gamma(t_1) \in L_1$, $\gamma(t_2) \in L_2$

Usually: $M = T^*N$, $L_1 = T_{q_1}^*N$, $L_2 = T_{q_2}^*N$

$(L_1 \times L_2) \cap L_I = \text{classical trajectories with } \begin{array}{l} q(t_1) = q_1 \\ q(t_2) = q_2 \end{array}$
Hamiltonian flow lines on T^*N generated by β

2) Scalar field (Euclidean).

• Space time $N =$ compact Riemannian oriented n -dimensional manifold

• Fields $F_N = \underbrace{\Omega^0(N)}_{\varphi} \oplus \underbrace{\Omega^{n-1}(N)}_p$

$$S_N[p, \varphi] = \int_N p \wedge d\varphi - \frac{1}{2} \int_N p \wedge *p - \int_N V(\varphi) dx$$

↑
Riemann volume form

$i: \partial N \hookrightarrow N$ inclusion of the boundary

$$\pi: F_N \rightarrow F_{\partial N} = \underbrace{\Omega^0(\partial N)}_{i^*(\varphi)} \oplus \underbrace{\Omega^{n-1}(\partial N)}_{i^*(p)}, \quad \theta \mapsto i^*(\theta)$$

$$\begin{aligned} \delta S' &= \int_N \delta p \wedge (d\varphi - *p) - \int_N (d p + V'(\varphi) dx) \delta \varphi + \\ &+ \int_{\partial N} i^*(p) \delta i^*(\varphi) \end{aligned}$$

• Euler-Lagrange equations, solutions form $EL_N \subset F_N$:

$$p = *d\varphi, \quad \Delta\varphi + V'(\varphi) = 0, \quad \Delta = d^*d = \text{Laplacian}$$

• Boundary 1-form

$$\alpha_{\partial N} = \int_{\partial N} p \delta\varphi, \quad p, \varphi - \text{boundary fields}$$

Symplectic form on boundary fields

$$\omega_{\partial N} = \int_{\partial N} \delta p \wedge \delta \varphi$$

$$* L_N = \pi(EL_N) = \{(p_0, \varphi_0) \in \Omega^0(\partial N) \oplus \Omega^{n-1}(\partial N) \mid \\ p_0 = i^*(\star d\varphi), \varphi_0 = i^*(\varphi); \Delta\varphi + V'(\varphi) = 0 \text{ in } N\}$$

when $V'(\varphi) = 0$ $L_N \cong \Omega^0(\partial N)$,

$$* C_N = \pi(\text{solutions to EL equations in a thin nbd of } \partial N) \cong F_{\partial N}$$

* symmetry = trivial; no gauge vector fields

Remark when $N = \Sigma \times [t_1, t_2]$ with

Minkowsky metric $(+ \dots + -)$ the above construction gives the usual Hamiltonian formulation of the scalar field.

3) BF theories:

- Space time: n -dimensional smooth, oriented, compact manifold with a principal G -bundle E over it.
- Fields \approx {pairs (A, B) , A is a connection on E , B is an $(n-2)$ -form on spacetime N with coefficients in $\tilde{E} = \mathfrak{g} \times_G E$ }

- The action:

$$S_N[A, B] = \int_N \text{tr}(B \wedge F(A))$$

- Euler-Lagrange equations, solutions = EL_N:

$$F(A) = 0, \quad d_A B = 0$$

- Boundary fields = $\{(a, b) \mid a \text{-connection on } E|_{\partial N}, b \in \Omega^{n-2}(\partial N, \tilde{E}|_{\partial N})\}$

$\pi: F_N \rightarrow F_{\partial N}$ is the pull-back to the boundary

- Variation of action \Rightarrow 1-form α on $F_{\partial N}$

$$d_{(a,b)} \left(\frac{\delta S}{\delta (a,b)} \right) = \int_{\partial N} \text{tr}(B \wedge \xi), \quad (\xi, \eta) \in T_{(a,b)} F_{\partial N}$$

Note: a is a connection (1-form locally); $\xi \in \Omega^1(\partial N, \tilde{E}|_{\partial N})$
 $\eta \in \Omega^{n-2}(\partial N, \tilde{E}|_{\partial N})$

Natural identification $F_{\partial N} = T^* \text{Conn}(E|_{\partial N})$

$a \in \text{Conn}(E|_{\partial N})$, $b \in T_a^* \text{Conn}(E|_{\partial N})$

If $\xi \in T_a \text{Conn}(E|_{\partial N})$, $b(\xi) = \int_N \text{tr}(b \wedge \xi)$
" $\Omega^1(\partial N, \tilde{E}|_{\partial N})$

• $C_{\partial N} = \{ \text{boundary values of solutions to EL in a nbd of } \partial N \} \subset F_{\partial N}$
" $\{b, a\}$, $F(a) = 0$, $d_a b = 0$

• $L_N = \{ \text{boundary values of solutions to EL equations in } N \} \subset C_{\partial N}$

Gauge symmetry

Bundle automorphisms and $B \mapsto B + d_A \theta$,

Infinitesimal (the action of the corresponding Lie algebra)

$$\delta_\lambda A = [A, \lambda] + d\lambda = d_A \lambda$$

$$\delta_\theta A = 0$$

$$\delta_\lambda B = [B, \lambda],$$

$$\delta_\theta B = B + d_A \theta$$

$$\delta_\lambda S_N = \int_N \text{tr}([B, \lambda] \wedge F(A) + B \wedge \delta_\lambda F(A)) = 0$$

$$\delta_\lambda F(A) = d_A \delta A = d_A (d_A \lambda) = [F(A), \lambda]$$

$$\delta_{\theta} S_N = \int_N \text{tr}(d_A \theta \wedge F(A)) = \int_{\partial N} \text{tr}(\theta \wedge F(a)) - \int_N \text{tr}(\theta \wedge d_A \hat{F}(A)) = \int_{\partial N} \text{tr}(\theta \wedge F(a))$$

S_N - λ -gauge invariant, θ -gauge invariant models the boundary term (vanishing on EL_N)

gauge vector fields on $F_{\partial N}$ are Hamiltonian

$$\delta_{\lambda} X = \{H_{\lambda}, X\}, \quad \delta_{\theta} X = \{H_{\theta}, X\}$$

$$H_{\lambda} = \int_{\partial N} \text{tr}(\lambda \wedge d_a b)$$

$$H_{\theta} = \int_{\partial N} \text{tr}(\theta \wedge F(a)),$$

Gives the moment map $\mu: F_{\partial N} \rightarrow (\text{gauge lie alg.})^*$

$$\mu^{-1}(0) = C_{\partial N} = \{(a, b) \mid d_a b = 0, F(a) = 0\}$$

The symplectic reduction of $C_{\partial N} = \mu^{-1}(0)/\text{gauge}$



Hamiltonian reduction

The space of leaves in the characteristic foliation of $C_{\partial N} = (C_{\partial N} / C_{\partial N}^{\perp})$ when $C_{\partial N}$ is linear

↑
symp. orthogonal

$$\mu^{-1}(0)/\text{gauge} = T^*(\text{gauge classes of flat connections})$$

$$= T^*(\text{Hom}(\pi_1(\partial N) \rightarrow G)/G)$$

4) Chern - Simons

- Space-time: smooth 3d with trivial principal G -bundle over it; G - compact, simple.

- Fields: connections $\equiv \Omega^1(N, \mathfrak{g})$

• The action:

$$S[A] = \frac{1}{2} \int_N \text{tr}(A \wedge dA) + \frac{1}{3} \int_N \text{tr}(A^3),$$

- Euler - Lagrange equations:

$$F(A) = 0, \quad F(A) = dA + \frac{1}{2} [A \wedge A]$$

Flat connections on $N \times G \rightarrow N$

- Boundary fields = {connections on $N \times G \rightarrow N$ } = $\Omega^1(\partial N, \mathfrak{g})$

- Boundary 1-form:

$$\alpha = \int_{\partial N} \text{tr}(a \wedge \delta a), \quad \left(\delta = \text{DR operator on the space of boundary fields} \right)$$

$$\omega = \delta \alpha = \int_{\partial N} \text{tr}(\delta a \wedge \delta a)$$

- $C_{\partial N} = \{ a \in \Omega^1(\partial N, \mathfrak{g}) \mid a = i^*(A), F(A) = 0, A \text{ is a connection on } N \times G \}$
 $=$ Flat connections on $\partial N \times G \rightarrow \partial N$

- $L_N =$ Flat connection on $\begin{matrix} \partial N \times G \\ \downarrow \\ \partial N \end{matrix}$ which continue to flat connections in $\begin{matrix} N \times G \\ \downarrow \\ N \end{matrix}$

Gauge transformations:

$$\delta_\lambda A = d_A \lambda$$

$$\delta_\lambda S_N[A] = \int_{\partial N} \text{tr}(\lambda F(A))$$

Gauge invariant on $EL_N = \text{Flat } G\text{-connections over } N$

Gauge action on boundary fields is Hamiltonian

$$\delta_\lambda X = \{H_\lambda, X\}$$

$$H_\lambda = \int_{\partial N} \text{tr}(\lambda F(A)), \quad \mu\text{-moment mapping}$$

Symplectic reduction of $C_{\partial N} = \mu^{-1}(0)/\text{gauge}$
Hamiltonian reduction

$$\text{Hom}(\pi_1(\partial N) \rightarrow G)/G = \mathcal{M}_{\partial N}^G$$

$$\text{Reduction of } L_N = \mathcal{L}_N \subset \mathcal{M}_{\partial N}^G$$

↑
gauge classes of flat G -connections over ∂N
which continue to flat G -connections over N .

5) Yang-Mills (Euclidean)

• Space time: Riemannian manifold,
n-dimensional, oriented, with a principal G-bundle
over it (we assume it is trivial).

• Fields: A = (connection on the principal G-bundle)
B = (g-valued (n-2)-form on N)

• The classical action:

$$S_N[A] = \int_N \text{tr}(B \wedge F(A)) + \frac{1}{2} \int_N \text{tr}(B \wedge *B)$$

• Euler-Lagrange equations

$$*B + F(A) = 0, \quad d_A B = 0$$

• Boundary fields $\Omega^1(\partial N, \mathfrak{g}) \oplus \Omega^{n-2}(\partial N, \mathfrak{g}) = F_{\partial N}$

The projection $F_N \rightarrow F_{\partial N}$ is the pull-back.

• Gauge transformations:

$$\delta_\lambda A = d_A \lambda$$