

Lecture 2. Semiclassical quantization of

Note Title

1/25/2012

first order classical theories

1) The mathematical framework of QFT:

Fix a space time category

A QFT assigns to each space time:

- a vector space $\mathcal{H}(\partial N)$
(the space of boundary states)
- a vector $z(N) \in \mathcal{H}(\partial N)$
(the "partition function")

These data should satisfy natural axioms:

$$(i) \quad \mathcal{H}(\emptyset) \simeq \mathbb{C}$$

$$(ii) \quad \mathcal{H}(\Sigma_1 \cup \Sigma_2) = \mathcal{H}(\Sigma_1) \otimes \mathcal{H}(\Sigma_2)$$

(iii) A nondegenerate pairing

$$\langle \cdot, \cdot \rangle_{\Sigma} : \mathcal{H}(\bar{\Sigma}) \otimes \mathcal{H}(\Sigma) \rightarrow \mathbb{C}$$

Here $\bar{\Sigma}$ is Σ with reverse orientation

(iv) An orientation reversing mapping $\sigma : \bar{\Sigma} \rightarrow \Sigma$

lifts to a \mathbb{C} -antilinear linear isomorphism

$\hat{\sigma} : \mathcal{H}(\bar{\Sigma}) \xrightarrow{\sim} \mathcal{H}(\Sigma)$ such that together with

$\langle \cdot, \cdot \rangle_{\Sigma}$ it induces a Hermitian form on $\mathcal{H}(\Sigma)$.

v) any orientation preserving isomorphism $\varphi: \Sigma_1 \rightarrow \Sigma_2$ lifts to a linear isomorphism $T_\varphi: \mathcal{H}(\Sigma_1) \rightarrow \mathcal{H}(\Sigma_2)$, which is compatible with $\langle \cdot, \cdot \rangle_\Sigma$ and

$$T_{\varphi \cup \psi} = T_\varphi \otimes T_\psi, \quad T_{\varphi \circ \psi} = T_\varphi \circ T_\psi$$

vi) If $M = M_1 \cup M_2$, $z_M = z_{M_1} \otimes z_{M_2}$

vii) If $\partial M = \partial' M \cup \Sigma \cup \Sigma'$, and $M_{\Sigma, f}$ is the result of gluing Σ to Σ' via $f: \Sigma \rightarrow \Sigma'$

$$z_{M_{\Sigma, f}} = \langle z_M \rangle_{\Sigma, f}$$



viii) $z_{\bar{M}} = \sigma_{\partial M} (z_M)$

where $\sigma_{\partial M}: \mathcal{H}(\partial M) \rightarrow \mathcal{H}(\bar{\partial M})$ is the lift of orientation reversing.

ix) let G be mapping between space times (bundle isomorphisms, diffeomorphisms, conformal maps etc. gauge transforms)

- The quantum field theory is G -invariant if
- each $g \in G$, $g: (N, E) \rightarrow (N', E')$ lift to $\hat{g}: \mathcal{H}(\partial N) \rightarrow \mathcal{H}(\partial N')$
- $z(N)$ and all operations are projectively G -invariant ($\hat{g} z(N) = z(N')$, etc.)

x) A quantum field theory quantizes a classical field theory if it comes as a family parametrized by \hbar such that as $\hbar \rightarrow 0$ it "reproduces" this classical field theory.

The concept of locality: "the theory can be glued from infinitesimally small pieces which interact only along common boundary."
(strictly speaking for this we should consider QFT for manifolds with corners.)

Remark 1. This framework was outlined by M. Atiyah for TQFT's and by G. Segal for CFT's.

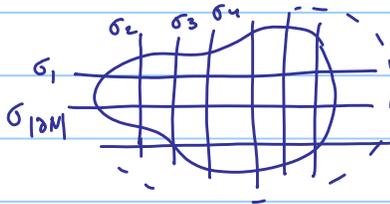
Remark 2. Assume $N = M \times [t_1, t_2]$, and $\partial M = \emptyset$
then $\partial N = \bar{M} \sqcup M$,
 $\mathcal{H}(\partial N) = \mathcal{H}(\bar{M}) \otimes \mathcal{H}(M)$ (completed)
 $\cong \text{End}(\mathcal{H}(M))$

$Z_{M \times [t_1, t_2]} \in \mathcal{H}(\partial N)$ is an operator in $\mathcal{H}(M)$

$Z_{M \times [t_1, t_2]}$ = the evolution operator from t_1 to t_2

Remark 3 This framework is very natural for statistical mechanics (on cell complexes)

Ising model:



$$\begin{aligned} \sigma_a &= \pm 1 \\ \mathcal{H}(\partial N) &= \bigoplus_{\sigma_1, \dots, \sigma_{|\partial N|} = \pm 1} \mathbb{C} e_{\sigma_1, \dots, \sigma_{|\partial N|}} \approx \\ &\approx (\mathbb{C}^2)^{\otimes |\partial N|} \end{aligned}$$

$$Z(N)_{\sigma_1, \dots, \sigma_{|\partial N|}} = \sum_{\substack{\tau \text{ - states of} \\ \text{the Ising model} \\ \tau|_{\partial N} = \sigma}} \exp(-\beta E(\tau))$$

Operations are obvious.

All local and non-local correlation functions can be obtained by cutting, gluing and composing $Z(N)$ with operators acting on $\mathcal{H}(\partial N)$.

2) The idea of geometric / path integral quantization of a classical 1st order field theory.

(i) To construct $\mathcal{H}(\partial N)$ fix a Lagrangian fibration

$$\begin{array}{ccc} F_{\partial N} \supset P^{-1}(b) & \text{Lagrangian submanifold} & \\ \downarrow & & \downarrow \\ P & & b \\ B_{\partial N} & \supseteq & \mathfrak{g} \end{array}$$

Define $\mathcal{H}(\partial N)$ as "the space of functions" on $B_{\partial N}$.

Examples: a) Classical Hamiltonian mechanics on $T^*\mathcal{N}$

$$I = [t_1, t_2], F_{\partial I} = T^*\mathcal{N} \times T^*\mathcal{N} \quad \text{fibers } T^*_{q_1}\mathcal{N} \times T^*_{q_2}\mathcal{N}$$

$$\downarrow$$

$$\mathcal{N} \times \mathcal{N}$$

$$\mathcal{H}(\partial I) = \text{End}(L_2(\mathcal{N})) \quad (\text{quantum mechanics})$$

b) BF theory: Nonreduced $F_{\partial N} = \Omega^1(\partial N) \oplus \Omega^{n-2}(\partial N)$
(abelian)

$$\downarrow$$

$$\Omega^1(\partial N)$$

Reduced:

$$T^*m_{\partial N}^G$$

$$\downarrow$$

$$m_{\partial N}^G$$

$$\mathcal{H}(\partial N)^G = \text{fnctns on } m_{\partial N}^G$$

c) Chern-Simons: M_Σ^G has no good Lagrangian fibration

$\mathcal{H}(\partial N) =$ homorphic sections of the geometric pre-quantization line bundle.

(ii) Given $b \in B_{\partial N}$ define $\int_{\mathcal{X}} e^{\frac{i}{\hbar} S_N(x)}$ in statistical field theory
 $\int_{\mathcal{X}} e^{-\beta S_N(x)}$ ↓
 $\mathcal{X} \in \pi^{-1}(p^{-1}(b))$ [$\int_{\mathcal{X}} e^{-\beta S_N(x)}$]

Here $\pi: F_N \rightarrow F_{\partial N}$, $p: F_{\partial N} \rightarrow B_{\partial N}$

Problems:

a) $\mathcal{H}_{\partial N}$ is the space of sections of the geometric quantization line bundle over $B_{\partial N}$.
Difficult to define when $F_{\partial N}$ is infinite dimensional.

Possible in TQFT's where reduced spaces are finite dimensional.

b) The integration over fields is hard to define when the space is infinite dimensional.

3) Standard ways to deal with these problems:

1) $\dim N = 1$, Quantum mechanics, $N = [t_2, t_1]$

$$\partial N = \overline{\{t_2\}} \cup \{t_1\}, \quad F_{\partial N} = (M, -\omega) \times (M, \omega)$$

when $M = T^*N$, natural choice of Lagrangian fibration:

$$\begin{array}{c} T^*N \\ \downarrow \\ N \end{array}$$

focus on the base of

$$\mathcal{H}(\partial I) = \text{End}(\mathcal{H}(\partial N)) = \text{End}(L_2(N))$$

$$Z(I) = \exp\left(\frac{i}{\hbar} \hat{H}(t_2 - t_1)\right) \in \mathcal{H}(\partial I)$$

is defined as the propagator for the Schrödinger equation.

- Semiclassical limit: WKB in the Schrödinger equation
- Path integral = Wiener integral (when $\hbar \rightarrow \frac{1}{\beta}$)

2) $\dim N > 1$, Constructive field theory (for statistical field theory)

- replace N by its metrized cell decomposition K with mesh δ_K
- Now F_K and $F_{\partial K}$ are finite dimensional and the naive construction (almost) make sense.

Difficult part is to pass to the limit $\delta_K \rightarrow 0$.

Semiclassical limit: $\hbar \rightarrow 0$ after $\delta_K \rightarrow 0$

(i.e. $\hbar \gg \delta_K$)

3) Perturbative approach (Feynman diagrams).

- discretize

- Replace the integral

$$z_K(\theta) = \int_{X \in \pi^{-1}(\bar{p}(\theta))} e^{\frac{i}{\hbar} S_K(x)} \mathcal{D}X$$

by its formal asymptotic $\hbar \rightarrow 0$

- pass to the limit $\delta_K \rightarrow 0$ in coefficients of the formal asymptotic

If this limit exists, the result is a formal power series which is expected to satisfy all properties of z_N .

Thm (T. Johnson - Freyd) This works in quantum mechanics.

Problem: when $\dim N > 1$ the limit $\delta_K \rightarrow 0$ in coefficients of the asymptotical expansion of z_N does not exist: ultraviolet divergencies.

In topological gauge theories UV problem does not exist, so we can try

a) construct perturbative TQFT

b) for Chern-Simons theory prove that

Reshetikhin - Turaev $\xrightarrow{\kappa \rightarrow \infty}$ perturbative Chern-Simons

Turaev - Viro \longrightarrow perturbative $\int_N \text{tr}(B \wedge F(A)) + \int_N \text{tr}(\Lambda^3 B)$

The goal: perturbative QFT for manifolds with boundary

1) Feynman diagrams: $V: \mathbb{R}^N \rightarrow \mathbb{R}$, analytic

$$V(x) = \sum_{n \geq 3} \frac{1}{n!} V^{(n)}(x), \quad V^{(n)}(x) = \sum_{i_1 \dots i_n} V_{i_1 \dots i_n}^{(n)} x^{i_1} \dots x^{i_n},$$

$$\int_{\mathbb{R}^N} e^{\frac{i}{2}(x, Bx) + \sum_{n \geq 3} \frac{i}{n!} V^{(n)}(x) \hbar^{\frac{n}{2}-1}} dx = \begin{cases} \text{Formal power} \\ \text{series expansion} \\ \text{in } \hbar \end{cases}$$

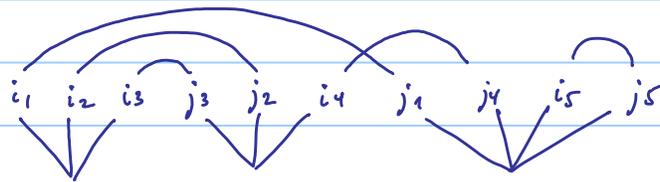
$$= (2\pi)^{\frac{N}{2}} \frac{e^{i\frac{\pi}{4} \text{sign}(B)}}{\sqrt{|\det(B)|}} \sum_{\Gamma} \frac{(i\hbar)^{-\chi(\Gamma)} F(\Gamma)}{|\text{Aut}(\Gamma)|},$$

Γ - graphs with 3, 4, ... valent vertices, $\chi(\Gamma)$ = its Euler characterist.

$|\text{Aut}(\Gamma)|$ = the number of automorphisms of Γ

$$F(\Gamma) = \sum_{\{i\} \subset \{1, \dots, n\}} \prod_{e \in m} (B^{-1})^{ieje} \prod_{\nu} V_{\{i_{\nu}, j_{\nu}\}}^{(m)}$$

Here we represent Γ as a pairing of a perfect matching m and stars of vertices:



$F(\Gamma)$ = contribution of Feynman diagram Γ

$$i \text{ --- } j \rightsquigarrow (B^{-1})^{ij}, \quad \begin{matrix} i_1 & \dots & i_n \\ \diagdown & & / \\ & \nu & \end{matrix} \rightsquigarrow V_{i_1 \dots i_n}^{(m)}$$

2) M = smooth compact, f = has finitely many isolated simple critical points, μ = measure on M

$$\int_M e^{i \frac{f(x)}{\hbar}} \mu \cong \sum_c \det(\mu_c) \hbar^{\frac{N}{2}} \int e^{i f(c, \hbar^{\frac{1}{2}} t)} dt$$

asymptotical equivalence

critical points \uparrow Jacobian $\underbrace{\quad}_{T_c M}$

$$N = \dim M$$

Formal power series described above with coefficients given by Feynman diagrams

Can not use in gauge theories where critical points of the action are not isolated

2) M - smooth, G acts on M , $f \in C^\infty(M)$

and $\mu = \text{measure on } M$ are G -invariant

$$\int_M e^{i \frac{f(x)}{\hbar}} \mu = |G| \int_{M/G} e^{i \frac{f(x)}{\hbar}} [\mu]$$

when $|G|$ is infinite l.h.s. is not defined.

Faddeev - Popov: G acts without stabilizers.

Let $\mathfrak{g} = \text{lie}(G)$, $\{e_a\}$ - basis in \mathfrak{g} ;

e_a act on M by the vector field $L_a(x) = \sum_i L_a^i(x) \partial_i$

Choose a section of $M \rightarrow M/G$

given by $\varphi^a(x) = 0$, i.e. $\varphi: M \rightarrow \mathfrak{g}$, s.t.

$\varphi^{-1}(0)$ intersect every G orbit only once

$$\int_{M/G} e^{i \frac{f(x)}{\hbar}} [\mu] = \hbar^{-n} \int_{\tilde{M}} e^{i \frac{f_{FP}(x, \bar{c}, \lambda)}{\hbar}} \mu d\bar{c} d\lambda$$

$$\tilde{M} = M \times \Pi \mathfrak{g} \times \Pi \mathfrak{g}^* \times \mathfrak{g}^*$$

$\Pi \mathfrak{g}$ - "odd" \mathfrak{g} , supermanifold $\begin{cases} \bar{c} & \text{coordinates on } \Pi \mathfrak{g} \\ c & \text{coordinates on } \Pi \mathfrak{g} \end{cases}$

$$f_{FP}(x, \bar{c}, \lambda) = f(x) - i\hbar \sum_{a,b} \bar{c}^a L_a^i(x) \partial_i \varphi^b(x) c_b + \sum_a \varphi^a(x) \lambda_a$$

If $f(x)$ has finitely many simple critical G -orbits

$$\int_{\tilde{M}} e^{\frac{i}{\hbar} f_{FP}(x, \bar{c}, \lambda)} \mu d\bar{c} d\lambda = \text{Const} \sum_{\substack{\text{critical} \\ \text{orbits } a \in \bar{\psi}^{-1}(0)}} \frac{\det(-iL(a))}{\sqrt{|\det(B(a))|}}$$

$$e^{i \frac{f(a)}{\hbar} + i \frac{\pi}{4} \text{sign}(B(a))} \sum_{\Gamma} \frac{(i\hbar)^{-\chi(\Gamma)} F(\Gamma)}{|\text{Aut}(\Gamma)|},$$



$$B^{-1}(a) = \begin{pmatrix} \partial_i \partial_j f(a) & \partial_i \varphi^b(a) \\ \partial_j \varphi^c(a) & 0 \end{pmatrix}^{-1}, \quad L(a)_c^b = L_c^i(a) \partial_i \varphi^b(a) \dots$$

Asymptotical expansion of $\int_{M/G} e^{\frac{i f(x)}{\hbar}} [\mu]$ is now given as a formal power series in \hbar with coefficients being Feynman diagrams.

3) BRST observation (Becchi, Rouet, Stora; Tyutin)

f_{FP} is a cocycle in the complex

$$C^\infty(M \times \mathfrak{g}^*) \wedge^\bullet (\mathfrak{g} \oplus \mathfrak{g}^*) \simeq \left\{ \begin{array}{l} \text{Chevalley-Eilenberg} \\ \text{complex for } \mathfrak{g} \\ \text{with coefficients} \\ \text{in } C^\infty(M) \end{array} \right\} \otimes \left\{ \begin{array}{l} \text{Koszul} \\ \text{complex} \\ \text{for } \mathfrak{g}^* \end{array} \right\}$$

$$Q_{\text{BRST}} = \underbrace{\sum_{a,i} c^a L_a^i(x) \frac{\partial}{\partial x^i} - \frac{1}{2} \sum_{abc} f_{bc}^a c^b c^c \frac{\partial}{\partial c^a}}_{Q_{\text{CE}}} + \underbrace{\sum_a \lambda_a \frac{\partial}{\partial \bar{c}_a}}_{Q_K}$$

$$Q_{\text{CE}}^2 = Q_K^2 = 0, \quad Q_{\text{CE}} Q_K = -Q_K Q_{\text{CE}}$$

$$Q_{\text{BRST}}^2 = 0 \quad (\text{supersymmetry})$$

$$f_{\text{FP}} = f + Q_{\text{BRST}} \left(\sum_a \varphi^a \bar{c}_a \right)$$

$$Q_{\text{BRST}} f_{\text{FP}} = 0$$

f = cohomology class of f_{FP}

A version of FP & BRST also work for symmetries described by involutive distributions on M when instead of

$$[L_a(x), L_b(x)] = f_{ab}^c L_c(x),$$

we have

$$[L_a(x), L_b(x)] = f_{ab}^c(x) L_c(x),$$

Does not work for non-involutive distributions.

Example: Poisson σ -model.

4) BV integrals work in non-involutive cases.