

Matrix models in Chern-Simons theory

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- Summary

Introduction to random matrix theory

Main definitions. Gaussian ensembles

- Let $H = (H_{jk})_{j,k=1}^N$ be a square $N \times N$ matrix with randomly distributed elements H_{jk} . This is a random matrix with respect to a probability distribution, defined by

$$P_\beta^N(H) \propto \exp(-\beta \text{Tr} V(H)).$$

- The first and most studied ensembles are the Gaussian ensembles, $V(H) = H^2$. It can be shown that the previous expression is automatically restricted to the form

$$P(H) = \exp(-a \text{Tr} H^2 + b \text{Tr} H + c), \quad a > 0.$$

if one postulates statistical independence of the matrix elements H_{jk} . There are three different ensembles defined, depending on the values of the parameter $\beta = 1, 2$ or 4 .

Introduction to random matrix theory

Orthogonal polynomial ensembles

- Diagonalization: for each matrix H there is a matrix U that maps it onto its eigenvalues. The Jacobian of the transformation is $J_\beta(\{x_i\}) = \prod_{i < j} |x_i - x_j|^\beta$. The resulting expression is

$$P(x_1, \dots, x_N) = C_N \prod_{i < j} |x_i - x_j|^\beta \exp\left[-\sum_{i=1}^N V(x_i)\right],$$

- The potential $V(x) = \log^2 x$ (log-normal weight function $\omega(x) = \exp(-\log^2 x)$) is at the center of most developments in this talk.
- The main relevant quantities are m-partial integrations over the previous N -dimensional probability density function

Introduction to random matrix theory

Orthogonal polynomials

- A central and powerful result in random matrix theory is that m -point correlation function can be computed from the two-point kernel as follows (simplest case of a Hermitian ($\beta = 2$) ensemble)

$$R_k(x_1, x_2, \dots, x_k) = \det [K_N(x_i, x_j)]_{1 \leq i, j \leq k}.$$

- Orthogonal polynomials method \Rightarrow explicit expressions for $K_N(x_i, x_j)$. Let $p_N(x) = c_N x^N + \dots$ the N th orthogonal polynomial associated to $e^{-V(x)}$, the two-point kernel is

$$\begin{aligned} K_N(x, y) &= e^{-\frac{(V(x)+V(y))}{2}} \sum_{k=0}^{N-1} p_k(x) p_k(y) \\ &= \frac{c_{N-1}}{c_N} \frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x-y} e^{-\frac{(V(x)+V(y))}{2}}. \end{aligned}$$

The Stieltjes-Wigert random matrix ensemble

Introduction to Chern-Simons theory

- We consider Chern-Simons theory on a three-manifold M and for a gauge group G , with action

$$S(A) = \frac{k}{4\pi} \int_M \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right),$$

where A is a connection on M .

- Witten showed in 1989, that the partition function of Chern-Simons theory

$$Z_k(M) = \int \mathcal{D}A e^{iS_{\text{CS}}(A)},$$

defines a topological invariant.

The Stieltjes-Wigert random matrix ensemble

Random matrix description. Partition functions.

- The partition function of CS theory on Seifert manifolds has simple expressions (M. Mariño, Comm. Math. Phys. 253, 25 (2004)). The simplest case is S^3 and gauge group $U(N)$

$$Z(S^3) = \int \prod_{i < j} \left(2 \sinh \frac{u_i - u_j}{2} \right)^2 \prod_{i=1}^N e^{-u_i^2/2g_s} \frac{du_i}{2\pi}.$$

- Thus, we have N -dimensional integral expressions for Chern-Simons partition functions whose expression resemble that of random matrix theory.
- One reason for these expressions: non-Abelian localization of CS path integral for Seifert manifolds (Beasley-Witten: hep-th/0503126)

Chern-Simons theory and the Stieltjes-Wigert matrix model

Partition function three-sphere $U(N)$ (M.T., Mod.Phys.Lett. A19, 1365 (2004))

- Ingredients: 1) Change of variables $e^{u_i} = x_i$ 2) Symmetry of the log-normal $\omega(qx) = \sqrt{q}x\omega(x)$ (when $\omega(x) = e^{-\log^2 x_i/2g_s}$), then

$$\begin{aligned} Z(S^3) &= \int \prod_{i < j} \left(2 \sinh \frac{u_i - u_j}{2} \right)^2 \prod_{i=1}^N e^{-u_i^2/2g_s} \frac{du_i}{2\pi} \\ &= (2\pi)^{-N} e^{\frac{-N^3 g_s}{2}} \int \prod_{i=1}^N dx_i e^{-\frac{\log^2(x_i)}{2g_s}} \prod_{i < j} (x_i - x_j)^2. \end{aligned}$$

- Last expression is the Stieltjes-Wigert matrix model. For the partition function computation, we actually only need the leading coefficients $p_i(x) = a_i x^i + \dots$, which are

$$a_j = q^{(j+1/2)^2} \left\{ (1-q) \dots (1-q^j) \right\}^{-1/2}.$$

Chern-Simons theory and the Stieltjes-Wigert matrix model

Partition function (M.T., Mod.Phys.Lett. A19, 1365 (2004))

- The partition function in terms of the orthogonal polynomials is:

$$\begin{aligned} Z &= \int \dots \int \prod_{i=1}^N \omega(x_i) dx_i \prod_{i < l} (x_i - x_l)^2 \\ &= \frac{N!}{\prod_{i=0}^{N-1} a_i^2} = N! a_0^{-2N} \prod_{i=1}^{N-1} \left(\left(\frac{a_{i-1}}{a_i} \right)^2 \right)^{N-i}. \end{aligned}$$

- Using the coefficients, we have $\left(\frac{a_{j-1}}{a_j} \right)^2 = q^{-4j} (1 - q^j)$, $a_0 = q^{1/4}$ and identifying $g_s = \frac{2\pi i}{k+N}$ (coupling constant with CS parameter) we finally find

$$Z(S^3) = e^{\frac{1}{4}i\pi N^2} (k+N)^{-N/2} \prod_{j=1}^{N-1} \left(2 \sin \frac{\pi j}{k+N} \right)^{N-j}.$$

Chern-Simons theory and the Stieltjes-Wigert matrix model

Quantum dimensions (Y. Dolivet and M.T. J. Math. Phys. 48, 023507 (2007))

- The Chern-Simons invariant of the unknot are quantum dimensions.
We showed

$$\begin{aligned}<\mathfrak{s}_\lambda(M)>_w &= \int [dM] \mathfrak{s}_\lambda(M) e^{-\frac{1}{2gs} \text{Tr}(\log M)^2} \\ &= q^{-n|\lambda| - \frac{1}{2}} C_\lambda^{U(n)} \mathcal{D}_\lambda,\end{aligned}$$

where $C_\lambda^{U(n)}$ is the Casimir of $U(N)$ and the last term

$$\mathcal{D}_\lambda \equiv \prod_{x \in \lambda} \frac{\lfloor n + c(x) \rfloor}{\lfloor h(x) \rfloor}$$

are the quantum dimensions.

Biorthogonal Stieltjes-Wigert model

YD and MT (J. Math. Phys. 48, 023507 (2007)), MT (J. Math. Phys. 51, 063509 (2010))

- We also have studied a biorthogonal version of the Stieltjes-Wigert model

$$Z^{P,Q} = \int \prod_i \frac{dx_i}{2\pi} e^{-\kappa^2 P^2 \log^2 x_i} \prod_{i < j} (x_i - x_j) (x_i^{P/Q} - x_j^{P/Q})$$

that appears when the manifold is a lens space, instead of S^3 .

- For this we had to find the biorthogonal version of the Stieltjes-Wigert polynomials ($k = P/Q$)

$$\int Y_n(x, k) x^{kj} \omega(x) dx = \alpha_n^{(k)} \delta_{n,j}, \quad \int Z_n(x, k) x^j \omega(x) dx = \beta_n^{(k)} \delta_{n,j}.$$

- The biorthogonal model with a Schur polynomial is relevant to the study of torus knots.

Summary

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- Random matrix models with a $V(x) = \log^2 x$ potential can be solved exactly, giving CS observables. Schur polynomials and combinatorics also needed in general.

Summary

- The relevant quantities associated to the probability distribution function of the eigenvalues of random matrices (like correlation functions, density of states), can be analytically computed with orthogonal polynomials.
- Random matrix models with a $V(x) = \log^2 x$ potential can be solved exactly, giving CS observables. Schur polynomials and combinatorics also needed in general.
- Main idea: Several "simple" gauge theories (e.g. topological) in low dimensions (3d and 2d) can be expressed in terms of random matrices and the observables of the theory computed and shown to have stochastic interpretations and connections with integrable systems.

Chern-Simons theory and the Stieltjes-Wigert matrix model

Density of states (S. de Haro and M.T., Nucl. Phys. B731, 225 (2005))

- The density of states can be computed exactly with the orthogonal polynomials using $\rho(x) = e^{-V(x)} \sum_{n=0}^{N-1} P_n(x)^2$. In the case $q \rightarrow 1$, the model tends to a Gaussian ensemble and the corresponding density of states tends to the well-known semi-circle law.

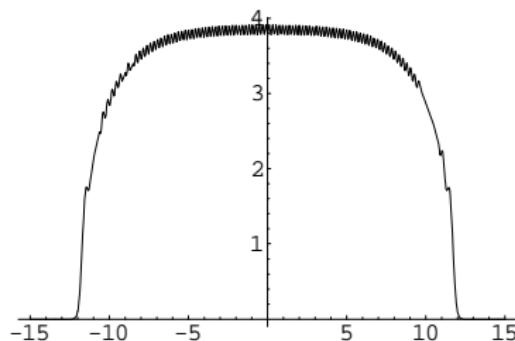
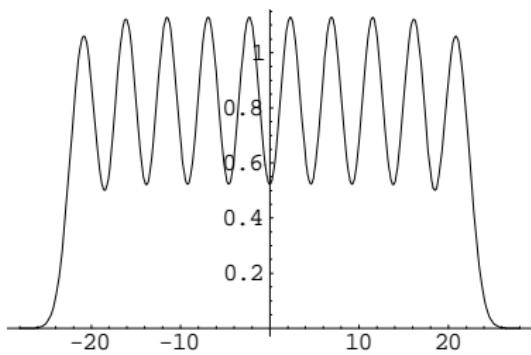


Figure: $q = 0.9$ and $N = 100$

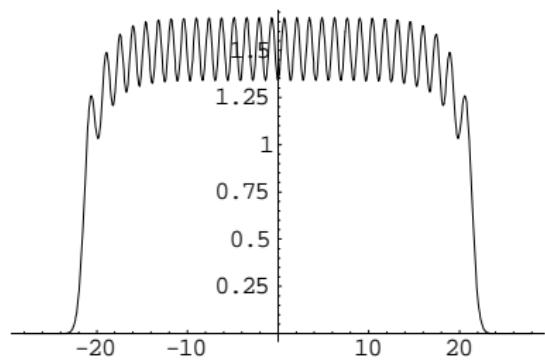
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Density of states (S. de Haro and M.T., Nucl. Phys. B731, 225 (2005))

- However, in contrast with ordinary (e.g. Gaussian) random matrix ensembles, the density of states shows a crystalline, oscillatory pattern.



$N = 10$ and $q = 0.3$



$N = 30$ and $q = 0.5$